Creating, Naming, and Justifying FRACTIONS

Fractions are one of the most difficult topics for students to learn in elementary school. In fact, children often find the topic of fractions to be nonsensical and mysterious. Consider, for example, the following exposition, which highlights how senseless fraction operations can appear:

> When we add or subtract fractions, we have to find a common denominator, but not when we multiply or divide. And once we get a common denominator, we add or subtract the numerators, but not the denominators, despite the fact that when we multiply, we multiply both the numerators and the denominators, and when we divide, we divide neither the numerators nor the denominators.

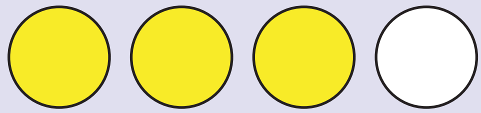
Although no teacher would say such a thing to children, this exposition points out an important fact: *Children are bound to find fraction computations arbitrary, confusing, and easy to mix up unless they receive help understanding what fractions and fraction operations mean.*

Research on teaching and learning fractions reveals that fractions have many possible meanings, including as part-whole relationships, quotients, or ratios (Witherspoon 1993). In fact, the Number and Operations Standard for grades 3–5 suggests that children need to develop all these meanings for fractions (NCTM 2000). Unfortunately, as children learn about fractions, they can easily lose sight of any of the foregoing meanings and instead see fractions merely as two whole numbers with a spatial relationship (that is, one is written above the other). Even pictures of fractions may not dislodge this whole-number concept of fractions. For example, in the picture of 3/4 found in **figure 1**, children may not see 3/4 at all, but rather three things shaded out of a total of four things. With this whole-number view of the picture, the fraction 3/4 is never really conceived of as a single number or quantity.

For students to develop meaningful concepts of fractions and fraction operations, they need to think of fractions in terms other than as just whole-number combinations. In this article, we suggest two powerful images for thinking about fractions that move beyond whole-number reasoning. These images are powerful for the following reasons. First, they make explicit the actions that children can perform on quantities to produce, compare, and operate on fractional parts. Children can use these images as tools to create and act on fractions. Second, these images provide ways for students to justify their fraction reasoning. In particular, they enable students to point to the pictures of fractions and fraction operations that they create and explain how they see the fractions in their pictures. Because these two images present ways to reason and talk about fractions, they can enable children to develop robust meanings for fractions and fraction operations.

Figure 1

A common representation of 3/4



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The Two Images: Iterating and Partitioning

To prompt and illustrate our discussion of the two images for fractions, first consider the following fraction problem:

> Suppose that the bar in **figure 2** represents 3/8. Create a bar that is equivalent to 4/3.

One possible solution would be to partition the 3/8 bar into 3 equal pieces, yielding 3 one-eighths. Once we have found the size of 1/8, we can easily make 1 whole by making 8 copies of the 1/8 and gluing them together. We can then create thirds by slicing the bar of size 1 into 3 equal pieces. The fraction 4/3 is merely 4 copies of 1/3, so we take our bar of size 1/3 and make 4 copies to produce 4/3.

This solution to creating 4/3 from 3/8 depends on two important actions: partitioning and iterating. The first action, partitioning, consists of creating smaller, equal-sized amounts from a larger amount. Often the larger amount is 1 whole, although it does not necessarily have to be so. We performed the action of partitioning when we cut 3/8 into 3 equal pieces to yield eighths and when we cut 1 into 3 equal pieces to yield thirds. The second action, iterating, consists of making copies of a smaller amount and combining them to create a larger amount. As with partitioning, the larger amount that we create is often 1 whole but does not necessarily have to be so. In our solution above, we iterated the 1/8 piece eight times to create 1 and the 1/3 piece four times to create 4/3.

The actions of iterating and partitioning have long been recognized as important to understanding and operating on fractions (Olive 2002; Pothier and Sawada 1990). However, the power that comes from embedding these actions into descriptions of fractions and fraction operations is seldom realized. Consider, for example, the fraction 1/8. What does this fraction mean? How do we describe it? Drawing on the actions of partitioning and iterating yields two possible descriptions:

- *Partitioning:* 1/8 is the amount we get by taking a whole, dividing it into 8 equal parts, and taking 1 of those parts.
- *Iterating:* 1/8 is the amount such that 8 copies of that amount, put together, make a whole (see **fig. 3**).

Children can use images as tools to create and act on fractions

Figure 2

A solution to making 4/3 of a bar when given 3/8 of a bar. (a) A bar of length 3/8. (b) Partition into 3 equal pieces to yield 1/8 pieces. (c) Iterate a 1/8 piece 8 times to find the length of 1 whole. (d) Partition the whole into 3 equal pieces to find the length of 1/3. (e) Iterate a 1/3 piece 4 times to get 4/3.

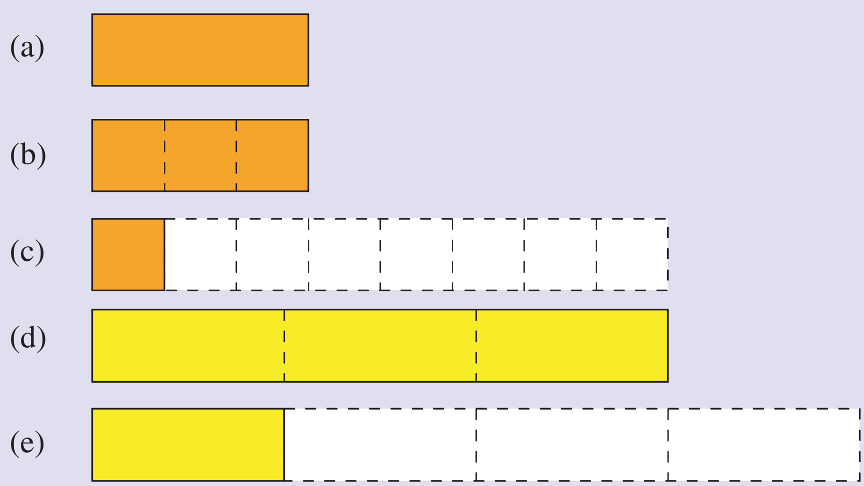
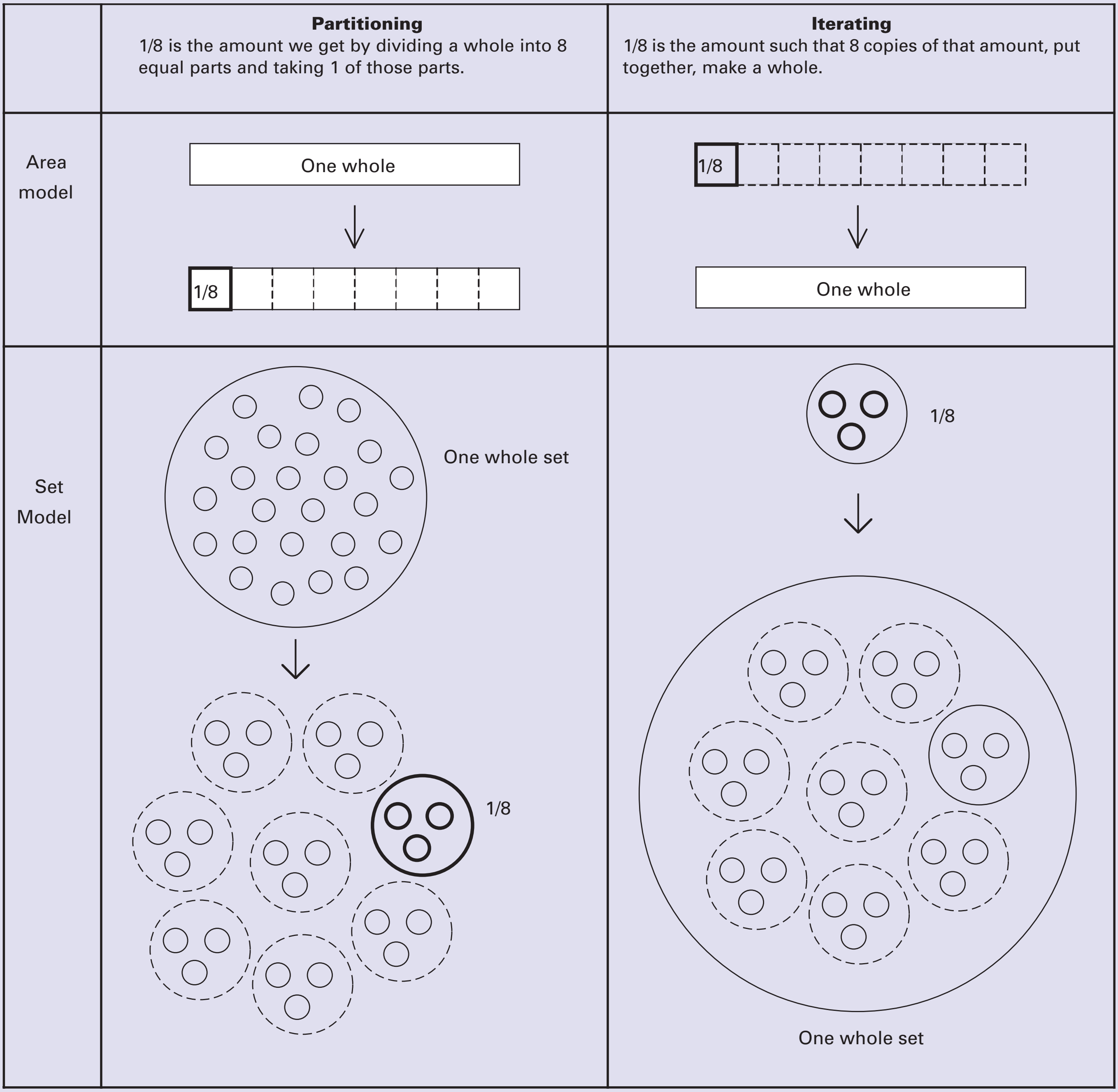


Figure 3

The fraction 1/8 from partitioning and iterating perspectives using both area and set models



Note that in the first description of 1/8, we have embedded the action that we could use to create 1/8. In the second description, we have embedded the action that we might use to confirm that the amount is actually 1/8. By conceptualizing 1/8 in both of these ways, we have automatically constructed ways of creating 1/8 and justifying that the amount we have created is actually 1/8. In other words, these ways of thinking about 1/8 provide tools for both acting and justifying.

The images of iterating and partitioning can also be used to describe all fractions, not just unit fractions (that is, fractions of the form 1/*n*, where *n* is a nonzero whole number). For example, the fraction 5/8 can be thought of as 5 one-eighths, where 1/8 is described either through iterating or partitioning, thus leading to two different descriptions for 5/8:

- *Partitioning:* 5/8 is 5 one-eighths, where 1/8 is the amount we get by taking a whole, dividing it into 8 equal parts, and taking 1 of those parts.
- *Iterating:* 5/8 is 5 one-eighths, where 1/8 is the amount such that 8 copies of that amount, put together, make a whole.

These two different ways of conceptualizing 5/8 can lead to a variety of ways of actually creating and justifying the quantity 5/8. For example, a child could partition a whole into 8 equal pieces to create eighths, and then take 5 of those eighths to make 5/8. When the child is asked to justify why the amount is 5/8, she can point to the process that she used to create the amount, namely, that each of the 5 parts are 1/8 because they were created by partitioning the whole into 8 equal parts, and the 5 parts together are 5 one-eighths, or 5/8. Alternatively, a child could create 5/8 by using iterating. He could do so through a repeatable guess-and-check process of guessing a length for 1/8, iterating until he has 8 pieces to compare it with the whole, modifying the initial guess, and repeating the process until he finds a length that when copied eight times produces the whole. Then he could create 5/8 by taking the 1/8 and making copies until he has 5 pieces. Once again, the justification that the created amount is actually 5/8 can be drawn from the process by which the amount was created: Each of the 5 parts is 1/8 because 8 copies of any 1 of the 5 parts produces the whole, and the 5 parts together are 5 one-eighths, or 5/8.

Although people typically have a preference for either partitioning or iterating, in reality powerful fraction reasoning involves both images. Without partitioning, the creation of smaller, equal-sized pieces is difficult; without iterating, the creation of larger pieces from smaller ones is difficult. As shown in the problem in **figure 2** involving the 3/8 bar, sophisticated reasoning about fractions often involves the use of both images in complementary ways. Likewise, compelling explanations and justifications of fraction reasoning frequently involve references to both images. In the following section, we show that these two images are different from whole-number images of fractions, thus letting us avoid many of the problems associated with whole-number images for fractions.

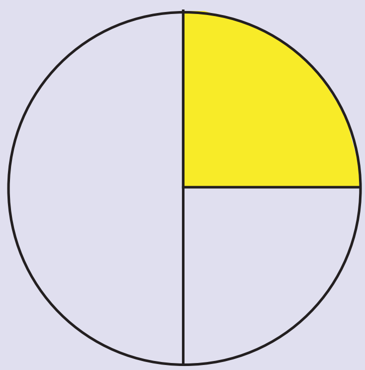
Why Not "Out Of"?

One of the most common ways of talking about fractions involves the phrase "out of," such as "5 out of 8" for 5/8. This phrase involves not only different language from that of iterating and partitioning but a different image as well. With an "out of" image, children see themselves presented with 8 things, then taking 5 from those 8 things. In this image, the numerator and denominator of the fraction are merely whole numbers. The 8 things are not thought of as eighths, nor is the 5 conceived of as 5 one-eighths. Furthermore, the actions that led to the creation of the parts (partitioning) or the creation of the whole (iterating) are never acknowledged, thus obscuring the relationship of the parts to the whole and the very actions that can be used to create and operate on fractions. Sole reliance on this image, without modifying it to incorporate images of iterating or partitioning, can cause children to have serious problems with fractions, including the following:

- Children may not realize that the pieces or groups associated with a particular unit fraction

Figure 4

One possible (wrong) representation for 1/3 when 1/3 is thought of as "1 out of 3"



must be the same size. For example, the shaded region in **figure 4** might be 1/3 when viewed from an "out of" perspective.
- Improper fractions may be nonsensical to children because they may think that a quantity that is more than the original amount is impossible. For example, 3/2 thought of as 3 out of 2 things is problematic, prompting the child to ask how she can take three things when she has only two things total.
- Children can easily lose sight of the referent whole and instead see the denominator as the referent whole. For example, if 3/4 is thought of as 3 out of 4, then 4 is the referent whole because that is where the 3 is taken from and what the 3 is compared with. This view is problematic because the referent whole may not be 4 at all. The fraction 3/4 may instead be referring to a fraction of a number other than 4, such as 3/4 of 2/5. When children lose sight of the referent whole, creating and naming fractional quantities can easily become confusing and uncertain.
- Children may have difficulty comparing fractions. They may think, for example, that because 2 candies are more than 1 candy, then "2 candies out of 8" is bigger than "1 candy out of 3."
- Children may never come to see fractions as being fundamentally different from whole numbers and thus may fail to understand fraction operations. For example, from an "out of" perspective, what does $3/4 \times 2/5$, or three-out-of-four times two-out-of-five, mean? Likewise, why doesn't $3/4 + 4/5 = 7/9$, since 3 out of 4 things plus 4 out of 5 things is obviously 7 out of 9 things, or 7/9?

The images of partitioning and iterating can address all these misconceptions

The images of partitioning and iterating can address all these misconceptions. Unit fractions of the same whole are always the same size because both images lead to the creation of equal-sized parts. Improper fractions, such as 3/2, can easily be interpreted as 3 one-halves. The referent whole retains relevancy because the images of iterating and partitioning make explicit the referent whole from which the fraction is created or compared. Fraction comparisons make sense because the fractional amounts are based on size relative to the referent whole, not on the number of pieces or parts they comprise. But perhaps the greatest benefit is that these two images can be very helpful in making sense of fraction operations, as discussed in the next section.

The Power of Partitioning and Iterating: Multiplication of Fractions

To illustrate how partitioning and iterating can be helpful in thinking about fraction operations, consider the operation of multiplication. For whole-number multiplication, the first number indicates the number of groups; the second number is the number of wholes, or ones, in each group; and the answer is the total amount of wholes, or ones. For example, 3×2 means 3 groups of 2 ones, which is a total of 6 ones. Thus, to solve a multiplication problem requires finding the total amount of ones that are in a certain number of groups of a certain size.

Next consider the fraction multiplication problem $3/4 \times 2/5$. Using the meaning for multiplication from above, $3/4 \times 2/5$ requires that we find how many ones are in 3/4 groups of 2/5, or 3/4 of 2/5. But what does 3/4 of 2/5 mean? It means 3 one-fourths of 2/5, where 1/4 of 2/5 is the amount we get from splitting 2/5 into 4 equal pieces (partitioning) and taking 1 of those pieces, or the amount that when copied and put together (iterating) gives us 2/5. The partitioning meaning is particularly helpful in actually creating a 1/4 piece of 2/5. All we have to do is find a way to cut 2/5 into 4 equal pieces. We can do so by splitting each of the 2 one-fifths in half. Then the resultant pieces are each 1/4 of 2/5 because they were created by cutting 2/5 into 4 equal pieces. To find 3/4 of 2/5, we need only take 3 of those 1/4 pieces of 2/5, as shown in **figure 5**.

The creation of 3/4 of 2/5, however, does not completely solve the problem, because we need to know what amount of the whole this quantity is. Unfortunately, from our picture, knowing what part of the whole we have created is difficult, because our picture is not entirely composed of the same-sized pieces. However, we can create the same-sized pieces throughout the whole by partitioning the remaining 3 one-fifths in the picture into two equal pieces each. These pieces are tenths of our whole. We can justify this action two different ways, either because the pieces were created through a process that cut our whole into 10 equal pieces (partitioning) or because any 1 of the pieces, when copied until we have 10 pieces, produces the whole (iterating). We have 3 of these tenths in our picture of 3/4 of 2/5, so the answer to 3/4 × 2/5 is three-tenths, or 3/10.

In this example, the images of partitioning and iterating are useful not only in operating on quantities but also in justifying what to call the resultant quantities after they are created. The decision of what to call quantities can often be difficult because many options are viable. In fact, when operating with fractions, being able to perceive multiple names for the same piece is often desirable. In the preceding example, the smallest-sized pieces in our pictures are thought of as halves (of 1/5), fourths (of 2/5), and tenths (of 1). The actions of partitioning and iterating enable us to conceptualize the smallest piece in three different ways, associate an appropriate name for each way of conceptualizing the piece, and justify why those names are appropriate.

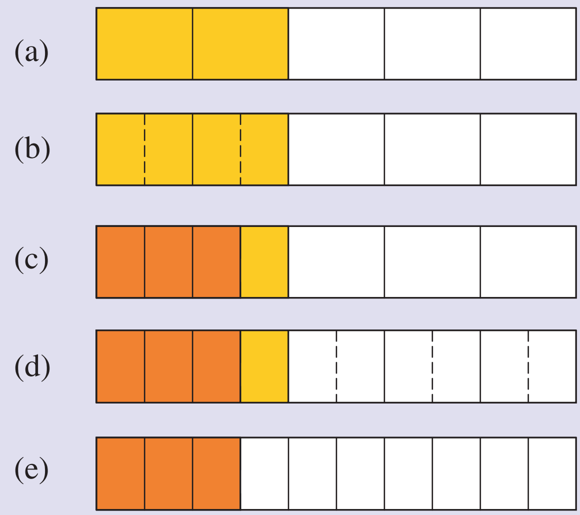
Implications for Teaching Fractions

Fortunately, you may need to make only minor changes in your fraction instruction to help your students develop images of partitioning and iterating. Children will naturally develop images of partitioning and iterating as they are given opportunities to create fractional amounts from wholes and wholes from fractional amounts, and asked to justify their reasoning. Particularly good beginning activities are those that require children to make a unit fraction from a whole, such as 1/4 from 1, or create a whole given a unit fraction, such as creating 1 given 1/3. Merely creating the amounts, however, is not enough; students must be given opportunities to talk about why the amounts they have created are correct and what unit fractions mean. During these discussions, you can elicit the images of partitioning and iterating that the children used in their initial investigations and link these images to the meaning of a unit fraction.

For example, consider the following exchange between Ms. M, an elementary school teacher, and Carla, a fifth-grade student. Ms. M has asked her students to create 1 given a bar of size 1/5. Rather than make 1 by iterating the bar to make 5 copies, Carla cuts the bar into 5 equal parts, making 1/25 instead. Ms. M is initially surprised by Carla's action, but then begins to question her about what it means to be 1/5. Ms. M discovers that for Carla, 1/5 is associated with the image of taking something and cutting it into 5 equal parts.

Figure 5

A solution for 3/4 × 2/5. (a) This figure depicts the given amount of 2/5. (b) To find 3/4 of 2/5, we first cut 2/5 into 4 equal pieces by cutting each of the 2 one-fifths in half. (c) We then take 3 of the 1/4 pieces from 2/5. (d) We need to cut our whole into pieces that are the same size as our 1/4 pieces of 2/5. To do so, we partition the remaining 3 one-fifths of our whole into 2 equal pieces each. These pieces are tenths of our whole. (e) We have 3 of these tenths pieces in our original picture of 3/4 of 2/5, so the answer to 3/4 × 2/5 is three-tenths, or 3/10.



Ms. M then asks Carla how she would create fifths of a piece of licorice so that she could share it fairly among herself and four of her friends. Carla responds that she would cut the licorice into 5 equal pieces. Ms. M then asks Carla to imagine that her friends have all eaten their pieces of licorice and that only Carla's uneaten piece remains. How would Carla show her brother how long the original rope of licorice was? Carla suggests drawing her piece 5 times, end to end, to recreate the original rope. As Ms. M and Carla talk about why this works, Ms. M guides Carla to discover a new meaning for 1/5 that is based on an iterating image.

The preceding example illustrates the usefulness of providing your students with fraction-creation and fraction-justification tasks. The tasks of creating a unit from the whole and then recreating the whole from the unit fraction naturally elicit the powerful images of iterating and partitioning. The process of justifying why the solution method works leads to a natural connection between these images and the meaning of the unit fraction. Once these meanings have been well established for unit fractions, your students can then move to creation and justification problems involving non-unit fractions by using the idea that a non-unit fraction is merely a certain number of copies of a unit fraction. You can guide your students in whole-class discussions to help establish classroom norms and practices that require them to justify their fraction reasoning in terms of these images. After such norms and practices are established, you can proceed with other fraction-modeling problems that you may have been using in past years, with the exception that you must frequently monitor your students' justifications to make sure that they continue to link their fraction reasoning with the powerful images of partitioning and iterating and not slip back into whole-number reasoning.

Language and Images

In this article, we have argued that partitioning and iterating are powerful images that can be used as tools for working with, making sense of, and justifying fractions and fraction operations. Although these two images are invariably tied to a certain collection of words and expressions, we have been careful to avoid prescribing exact phrases or verbs to describe these images. We believe that the images, not the specific language used to describe the images, should receive the most emphasis. Thus, many phrases, including "cut evenly," "split equally," or "separate into equal parts," can be used to describe partitioning. Likewise, iterating can be described by such phrases as "making copies" or "repeating end to end." In fact, we anticipate that most teachers will choose not to use the terms *partitioning* and *iterating* when discussing these two strategies with children but will instead adopt the idiosyncratic terms that the children use to describe partitioning and iterating during their initial explorations of fractions.

With that said, we must caution that teachers still need to attend closely to the language that children use as they express their mathematical reasoning. The expressions that children (and adults, for that matter) use often involve underlying images, and these images vary in appropriateness and usefulness. The phrase "out of," as shown previously, is inherently problematic because the image typically associated with this language is not helpful in reasoning about many different concepts. Likewise, the expression "over," such as "3 over 4" to describe 3/4, is problematic for the same reason. Because not all images are equal in their usefulness, teachers must listen carefully to children's language for clues about the children's underlying images, and then decide which images they wish to nurture and which images they wish to avoid. For fractions, we recommend that teachers nurture the images of partitioning and iterating and avoid images that rely on whole-number thinking.

References

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