# Mental Math

**In**

# Mathematics 2

NOVA SCOTIA
Education
English Program Services

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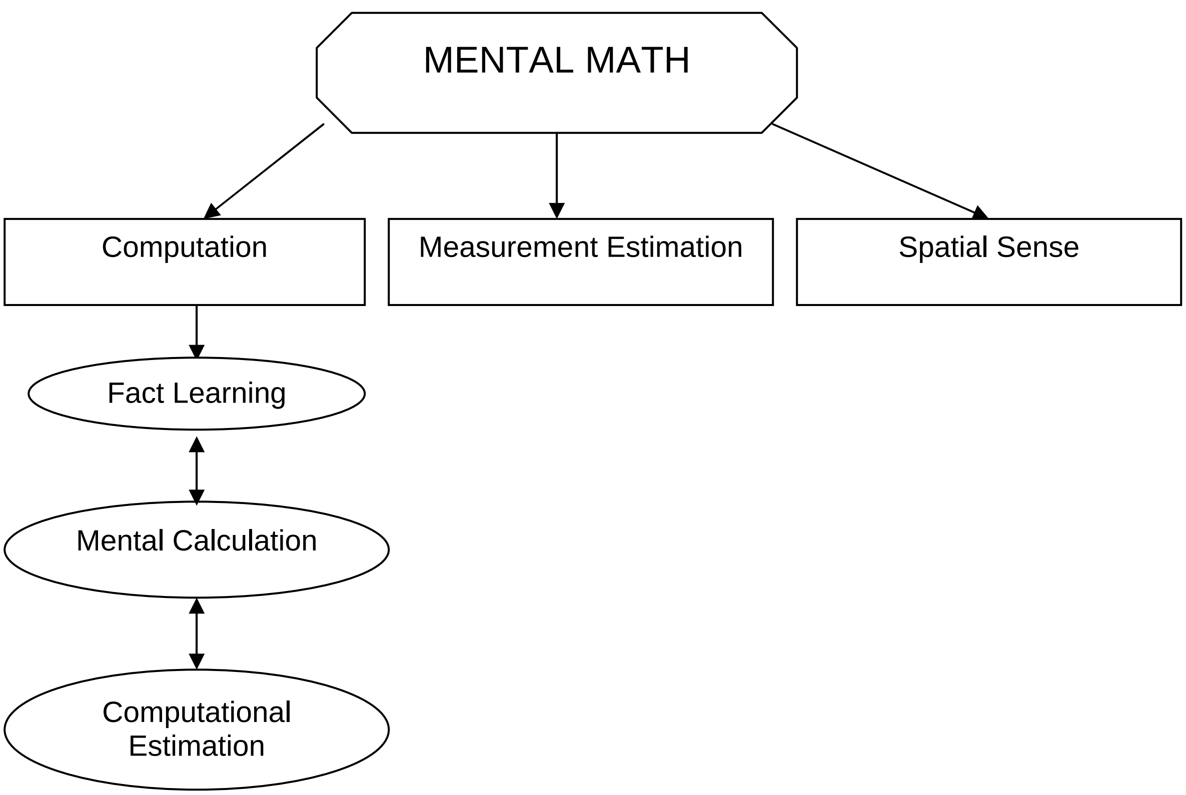
# Contents

# Introduction

Welcome to your grade-level mental math booklet. After the Department of Education released the *Time to Learn* document in which at least five minutes of mental math was required daily in grades 1 – 9, it was decided to support teachers by clarifying and outlining the specific mental math expectations at each grade. Therefore, grade-level booklets for computational aspects of mental math were prepared and released in draft form in the 2006–2007 school year. Building on these drafts, the current booklets describe the mental math expectations in computation, measurement, and geometry at each grade. These resources are supplements to, not replacements of, the Atlantic Canada mathematics curriculum. You should understand that the expectations for your grade are based on the full implementation of the expectations in the previous grades. Therefore, in the initial years of implementation, you may have to address some strategies from previous grades rather than all of those specified for your grade. It is critical that a school staff meets and plans on an on-going basis to ensure the complete implementation of mental math.

## Definitions

In the mathematics education literature, there is not consensus on the usage of some of the words and expressions in mental math. In order to provide uniformity in communication in these booklets, it is important that some of these terms be defined. For example, the Department of Education in Nova Scotia uses the term *mental math* to encompass the whole range of mental processing in all strands of the mathematics curriculum. *Mental math* is broken into three categories in the grade-level booklets: *mental computation*, *measurement estimation*, and *spatial sense*. *Mental computation* is further broken down into *fact learning*, *mental calculation*, and *computational estimation*.



*Fact learning* refers to the acquisition of the 100 number facts related to the single digits 0 to 9 for each of the four operations. When students know these facts, they can quickly retrieve them from memory (usually in 3 seconds or less). Ideally, through practice over time, students will achieve automaticity; that is, they will have instant recall without using strategies. *Mental calculation* refers to getting exact answers by using strategies to do the calculations in one's head, while *computational estimation* refers to getting approximate answers by using strategies to do calculations in one's head.

While each category in computations has been defined separately, this does not suggest that the three categories are entirely separable. Initially, students develop and use strategies to get quick recall of the facts. These strategies and the facts themselves are the foundations for the development of other mental calculation strategies. When the facts are automatic, students are no longer employing strategies to retrieve them from memory. In turn, the facts and mental calculation strategies are the foundations for computational estimation strategies. Actually, attempts at computational estimation are often thwarted by the lack of knowledge of the related facts and mental calculation strategies.

*Measurement estimation* is the process of using internal and external visual (or tactile) information to get approximate measures, or to make comparisons of measures, without the use of measurement instruments.

*Spatial sense* is an intuition about shapes and their relationships, and an ability to manipulate shapes in one's mind. It includes being comfortable with geometric descriptions of shapes and positions.

## Rationale for Mental Math

In modern society, the development of mental skills needs to be a major goal of any mathematics program for two major reasons. First, in their day-to-day activities, most people's computational, measurement, and spatial needs can be met by having well developed mental strategies. Secondly, because technology has replaced paper-and-pencil as the major tool for complex tasks, people need to have well developed mental strategies to be alert to the reasonableness of the results generated by this technology.

# PART 1

## Computation

# The Implementation of Mental Computation

## General Approach

In general, a computational strategy should be introduced in isolation from other strategies, a variety of different reinforcement activities should be provided until it is mastered, the strategy should be assessed in a variety of ways, and then it should be combined with other previously learned strategies.

### A. Introducing a Strategy

The approach to highlighting a computational strategy is to give the students an example of a computation for which the strategy would be useful to see if any of the students already can apply the strategy. If so, the student(s) can explain the strategy to the class with your help. If not, you could share the strategy yourself. The explanation of a strategy should include anything that will help students see the pattern and logic of the strategy, be that concrete materials, visuals, and/or contexts. The introduction should also include explicit modeling of the mental processes used to carry out the strategy, and explicit discussion of the situations for which the strategy is most appropriate and efficient. Discussion should also include situation for which the strategy would not be the most appropriate and efficient one. Most important is that the logic of the strategy should be well understood before it is reinforced; otherwise, it's long-term retention will be very limited.

### B. Reinforcement

Each strategy for building mental computational skills should be practised in isolation until students can give correct solutions in a reasonable time frame. Students must understand the logic of the strategy, recognize when it is appropriate, and explain the strategy. The amount of time spent on each strategy should be determined by the students' abilities and previous experiences.

The reinforcement activities for a strategy should be varied in type and should focus as much on the discussion of how students obtained their answers as on the answers themselves. The reinforcement activities should be structured to insure maximum participation. At first, time frames should be generous and then narrowed as students internalize the strategy. Student participation should be monitored and their progress assessed in a variety of ways to help determine how long should be spent on a strategy.

After you are confident that most of the students have internalized the strategy, you need to help them integrate it with other strategies they have developed. You can do this by providing activities that includes a mix of number expressions, for which this strategy and others would apply. You should have the students complete the activities and discuss the strategy/strategies that could be used; or you should have students match the number expressions included in the activity to a list of strategies, and discuss the attributes of the number expressions that prompted them to make the matches.

**Language**

Students should hear and see you use a variety of language associated with each operation, so they do not develop a single word-operation association. Through rich language usage students are able to quickly determine which operation and strategy they should employ. For example, when a student hears you say, "Six plus five", "Six and five", "The total of six and five", "The sum of six and five", or "Five more than six", they should be able to quickly determine that they must add 6 and 5, and that an appropriate strategy to do this is the *double-plus-one* strategy.

**Context**

You should present students with a variety of contexts for each operation in some of the reinforcement activities, so they can apply operations and strategies to situations found in their daily lives. By using contexts such as measurement, money, and food, the numbers become more real to the students. Contexts also provide you with opportunities to have students recall and apply other common knowledge that should be well known.

For example, when a student hears you say, “How many days in two weeks?” they should be able to recall that there are seven days in a week and that double seven is 14 days.

**Number Patterns**

You can also use the recognition and extension of number patterns can to reinforce strategy development. For example, when a student is asked to extend the pattern “30, 60, 120, …,”, one possible extension is to double the previous term to get 240, 480, 960. Another possible extension, found by adding multiples of 30, would be 210, 330, 480. Both possibilities require students to mentally calculate numbers using a variety of strategies.

**Examples of Reinforcement Activities**

Reinforcement activities will be included with each strategy in order to provide you with a variety of examples. These are not intended to be exhaustive; rather, they are meant to clarify how language, contexts, common knowledge, and number patterns can provide novelty and variety as students engage in strategy development.

### C. Assessment

Your assessments of computational strategies should take a variety of forms. In addition to the traditional quizzes that involve students recording answers to questions that you give one-at-a-time in a certain time frame, you should also record any observations you make during the reinforcements, ask the students for oral responses and explanations, and have them explain strategies in writing. Individual interviews can provide you with many insights into a student’s thinking, especially in situations where pencil-and-paper responses are weak.

Assessments, regardless of their form, should shed light on students’ abilities to compute efficiently and accurately, to select appropriate strategies, and to explain their thinking.

## Response Time

Response time is an effective way for you to see if students can use the computational strategies efficiently and to determine if students have automaticity of their facts.

For the facts, your goal is to get a response in 3-seconds or less. You would certainly give students more time than this in the initial strategy reinforcement activities, and reduce the time as the students become more proficient applying the strategy until the 3-second goal is reached. In subsequent grades, when the facts are extended to 10s, 100s and 1000s, you should also ultimately expect a 3-second response.

In the early grades, the 3-second response goal is a guideline for you and does not need to be shared with your students if it will cause undue anxiety.

With mental calculation strategies and computational estimation, you should allow 5 to 10 seconds, depending upon the complexity of the mental activity required. Again, in the initial application of these strategies, you would allow as much time as needed to insure success, and gradually decrease the wait time until students attain solutions in a reasonable time frame.

## Integration of Strategies

After students have achieved competency using one strategy, you should provide opportunities for them to integrate it with other strategies they have learned. The ultimate goal is for students to have a network of mental strategies that they can flexibly and efficiently apply whenever a computational situation arises. This integration can be aided in a variety of ways, some of which are described below.

You should give them a variety of questions, some of which could be done just as efficiently by two or more different strategies and some of which are most efficiently done by one specific strategy. It is important to have a follow-up discussion of the strategies and the reasons for the selection of specific strategies.

You should take every opportunity that arises in regular math class time to reinforce the strategies learned in mental math time.

You should include written questions in regular math time. This could be as journal entry, a quiz/test question, part of a portfolio, or other assessment for which students will get individual feedback. You might ask students to explain how they could mentally compute a given question in one or more ways, to comment on a student response that has an error in thinking, or to generate sample questions that would be efficiently done by a specified strategy.

# A. Addition — Fact Learning

At the beginning of grade 2, you should review addition facts and strategies that your students learned in grade 1. This should include recall of facts to 10 that would have been learned largely through visualization of ten frames, double facts that would have been learned through association with real world objects, and plus-one facts that would have been learned through association with a call for the next number. You will build on their previously learned facts until they are able to recall all addition facts (100) with a three-second, or less, response by the end of grade 2.

Students are more likely to be successful learning mental fact strategies if (a) they have reached the stage of development in addition where they quite naturally, with no prompting, count on from the larger number; (b) they quickly respond correctly to sums involving 10, such as 10 + 6 or 4 + 10; and (c) they are comfortable partitioning small numbers and “seeing” numbers within numbers, such as a 5 or a 7 within an 8. Consequently, you should try to ensure that most of your students have these three skills/concepts before launching into extensive work with fact strategy learning. In the early part of the year these skills/concepts should be the focus in the study of addition and subtraction.

Provide each student with an addition chart in an array format. As each cluster of facts is introduced, students can fill them in their charts. If they use a different colour pencil for each cluster, their charts will be colour-coded for the strategies. This gradual development of the addition chart helps students (and their parents) focus on the facts that are being learned, and can provide a motivational tool. Students need to achieve the desired response time for each cluster before moving on; otherwise, they will get overwhelmed and frustrated.

You are provided with a possible sequence of strategies, as well as the cluster of facts related to each of these strategies. Some facts could be included in more than one cluster; however, this sequence only lists the new facts for each strategy. As students fill in their addition charts as each new strategy is introduced, they could circle those facts that are already on the chart but which could also be done by this new strategy.

In this provided sequence, all facts involving zeros are left until 51 other facts are mastered: this approach is partly for motivational reasons because students get to add 19 relatively easy facts to their charts to get a total of 70. This approach is also suggested to help counter a common misconception that causes many students to make errors with these facts: they over generalize that addition involves an action that results in a larger quantity; thus, they are reluctant to give a response that shows no change. As well, addition sentences with zeros are not likely encountered in response to story problems or in situations in their everyday lives.

While each cluster of facts will have suggested reinforcements, here is one possible reinforcement activity that could be used for each cluster of facts and for the integration of clusters. Prepare a large poster (or overhead acetate) of pictures of ten articles, each with a different prominent price tag ranging from 0¢ to 9¢ or $0 to $9. For example, you could have the following pictures: Piece of paper - 0¢, paper clip - 1¢, file card - 2¢, balloon - 3¢, candy - 4¢, pencil - 5¢, eraser - 6¢, pen - 7¢, marker - 8¢, and ruler - 9¢. Ask students how much they would pay for two articles that you name, being sure to name combinations that fit the strategy cluster(s) you are reinforcing. For example, after students know double, plus-one, and 1-apart facts, you could ask them how much they would pay for: a) a ruler and a paper clip, b) an eraser and pencil, c) two balloons, d) a pen and marker, e) a paper clip and a candy, f) a file card and a balloon, etc.

## Doubles Facts (Review)

*The nine facts: 1 + 1, 2 + 2, 3 + 3 4 + 4, 5 + 5,6 + 6, 7 + 7, 8 + 8, 9 + 9*

These facts were learned in grade 1 through association. The table below gives examples of possible associations. Use these or others with which students can strongly identify.

| Double Fact | Possible Association |
|---|---|
| 1 + 1 | Number of tires on two unicycles |
| 2 + 2 | Number of tires on two bicycles |
| 3 + 3 | Number of tires on two tricycles<br>Number of sides on two triangles<br>Six pack of pop |
| 4 + 4 | Number of tires on two cars<br>Number of sides on two squares |
| 5 + 5 | Number of fingers on two hands |
| 6 + 6 | Dozen eggs in a carton |
| 7 + 7 | Number of days in two weeks |
| 8 + 8 | Number of crayons in two rows in a box<br>Number of legs on two octopuses |
| 9 + 9 | Number of tires on an 18-wheeler truck |

**Examples**

For 5 + 5, think: The number of fingers on two hands is 10.

For 9 + 9, think: The number of tires on an 18-wheeler is 18.

**Examples of Some Practice Items**

4 + 4

5 and 5

8 kg greater than 8 kg

7¢ plus 7¢

$3 more than $3

Double 9 minutes

## Plus Ones Facts (Review)

*The 16 facts: 2 + 1, 3 +1, 4 + 1, 5 + 1, 6 + 1, 7 + 1, 8 + 1, 9 + 1 and their commutative pairs 1 + 2, 1 + 3, 1 + 4,*
*1 + 4, 1 + 6, 1 + 7, 1 + 8, 1 + 9 (Note: 1 + 1 was already a double but could also be thought of as next number.)*

These strategies were learned in grade 1 through association with a call for the next number. That is, whenever students see a 1 in an addition phrase, they should look at the other number and think: What number comes after this number? To help students visualize this, they could use ten-frames for all the numbers from 1 to 9 and notice the one-dot difference between consecutive numbers. They could also make Unifix Cube towers for the numbers 1 to 9, and notice that the addition of one cube produces the next number tower. Relating this addition of 1 to the next number is also very clear on a number line.

Time should be spent first with the set of ___ + 1 facts, and then with the set of 1 + ___ facts, being careful that students are convinced about the commutative nature of addition. Finally, mix both sets. When you are satisfied that students have a 3-second, or less, response time for these facts, integrate them with the double facts and reinforce all 25 facts.

**Examples**

For 6 + 1, think: The number after 6 is 7.

For 1 + 4, think: The number after 4 is 5.

**Examples of Some Practice Items**

3 + 1 =

5 and 1

1 more than 4

$1 plus $6

The total of 9 and 1

One hour after 6:00

## 1-Apart (Near Doubles) Facts (New)

*The 14 facts: 2 + 3, 3 + 4, 4 + 5, 5 + 6, 6 + 7, 7 +8, 8 + 9 and their commutative pairs 3 + 2, 4 + 3, 5 + 4, 6 + 5,*
*7 + 6, 8 + 7, 9 + 8. (Note: 1 + 2 and 2 + 1 were already included in the Plus One facts but could also be thought of as 1-Aparts.)*

This strategy is a combination of the *doubles facts* and the *plus-one facts*. It involves doubling the smaller number and adding one. (Some students might double the larger and subtract 1.) Students need to be convinced that the larger number can be partitioned without changing the sum. For example, they need to see that for 4 + 5, the 5 can be partitioned into 4 and 1, and that the 4 can be combined with the other 4 before adding the 1. You could have students create Unifix Cube towers for numbers 2 to 9, have them take two consecutive number towers, and take one cube off the larger tower so they can see the double towers and the 1. In symbols: $4 + 5 = 4 + (4 + 1) =$
$(4 + 4) + 1 = 8 + 1 = 9$.

When you are confident that students have the necessary response time for these *1-apart facts,* integrate them with the *double facts* and the *plus-one facts*, and reinforce these 39 facts.

**Examples**

For 2 + 3, think: double 2 is 4 and the next number is 5; or 2 and 2 is 4, and plus 1 is 5.

For 8 + 7, think: double 7 is 14 and the next number is 15; or 7 and 7 is 14 , and plus 1 is 15.

**Examples of Some Practice Items**

8 + 9 =

The sum of 7 cookies and 8 cookies

The total of 6 and 5

4 cm and 5 cm

$3 more than $2

6 girls. 7 boys. How many children?

## Plus Two Facts (New)

*The 12 facts: 4 + 2, 5 + 2, 6 + 2, 7 + 2, 8 + 2, 9 + 2 and their commutative pairs 2 + 4, 2 + 5, 2 + 6, 2 + 7, 2 + 8, 2 + 9*. *(Note: 1 + 2, 2 + 2, 3 + 2, 2 + 1, and 2 + 3 have already been included in other strategies but could also be thought of as next even or odd numbers.)*

This strategy involves associating the addition of 2 with skip counting by 2, or with getting the next even or odd number. You could help students get ready for this strategy by conducting skip-counting chants starting at different numbers. After reinforcing these 12 facts until the response time is achieved, integrate then with the other 39 through reinforcement of all 51 facts.

**Examples**

For 5 + 2, think: 5 is an odd number and the next odd number is 7; or Skipping 2 starting at 5 is 7.

For 6 + 2, think: 6 is an even number and the next even number is 8; or Skipping 2 starting at 6 is 8.

**Examples of Some Practice Items**

2 + 8

$2 more than $5

7 plus 2

9¢ and 2¢

4 increased by 2

2 dogs and 6 cats. How many pets?

## Plus Zero Facts (New)

*The 19 facts: 0 + 0. 1 + 0, 2 + 0, 3 + 0, 4 + 0, 5 + 0, 6 + 0, 7 + 0, 8 + 0, 9 + 0 and their commutative pairs 0 + 1, 0 + 2, 0 + 3, 0 + 4, 0 + 5, 0 + 6, 0 + 7, 0 + 8, 0 + 9.*

This strategy involves the association of adding zero with making no change to the other addend. This should be introduced by reference to story problems that would be represented by the addition of zero. For example, I had 5 cookies. I met my friend who would not give me any more cookies. How many cookies did I have after I met my friend? (Telling the same story with the friend giving 3 cookies and writing the appropriate number sentence, then 2 cookies, and 1 cookie before the 0 cookies would help underscore the addition sentence with zero.) Stories, such as this one, are not likely to have been encountered before, and while students probably think of them as silly, they do make the point about the role of zero in addition.

When reinforced in isolation, these facts are not problematic; however, when integrated with other facts, some students treat them as if they were plus-one facts because they over generalize that addition is an action that makes a larger quantity and are reluctant to record no change as a result. Consequently, you will probably not have to spend much time reinforcing these 19 facts in isolation; rather, more time should be spent reinforcing the total of 70 facts.

**Examples**

For 5 + 0, think: adding zero will make no change to 5, so the answer is 5.

For 0 + 7, think: adding zero will make no change to 7, so the answer is 7.

**Examples of Some Practice Items**

9 + 0

0 more than 3

$6 collected on Monday and $0 on Tuesday. How much altogether?

0 and 0 more

0 oatmeal cookies and 5 chocolate chip cookies. How many cookies?

8 increased by zero

## Make-10 Facts (New)

*The 10 facts involving 9: 9 + 3, 9 + 4, 9 + 5, 9 + 6, 9 + 7 and their commutative pairs 3 + 9, 4 + 9, 5 + 9, 6 + 9,*
*7 + 9; and the 8 facts involving 8: 8 + 3, 8 + 4, 8 + 5, 8 + 6 and their commutative pairs 3 + 8, 4 + 8, 5 + 8, 6 + 8.*

This strategy involves taking 1 or 2 from the one addend to make the 9 or 8 addend a 10, and then adding this 10 to what was left from the other addend. Students need to be convinced that adding 9 or 8 is as easy as adding 10. Start with the facts that involve 9s, and reinforce them. Once the response time is achieved for these facts, integrate them with the other 70 facts, and then move on to isolate the facts involving 8s. Once the response time for these facts is achieved, a total of 88 facts can then be reinforced.

To help students understand the logic of this strategy, provide them with ten-frame cards for 1 to 9. Ask them to place the 9-card on the left and to place the 1 to 8 cards in a column on the right. As you give each fact, students should place the 9-card next to the other appropriate card and visually move one dot to complete the 9-card. For example, for 9 + 6, students should visualize one dot from the 6-card moving onto the 9-card so the cards become a 10-card and a 5-card.

Instead of this *Make-10* strategy, some students may naturally add 10, instead of 9, to the other number and then adjust the answer by subtracting 1. It is not necessary that students know the name of this strategy; however, it is the *Compensation* strategy.

**Examples**

For 9 + 7, think: Take 1 from the 7 to make the 9 a 10, and then add 10 and 6 to get 16.

For 8 + 4, think: Take 2 from the 4 to make the 8 a 10, and then add 10 and 2 to get 12.

**Examples of Some Practice Items for 9s:**

9 + 4

$3 more than $9

A mass of 9 kg increased by 5 kg

6¢ and 9¢

8 girls and 9 boys. How many children?

9 cookies and 7 bags of chips. How many treats?

**Examples of Some Practice Items for 8s:**

8 + 5

$3 more than $8

8 days of school and 4 holidays. How many days altogether?

6 and 8

7 dogs and 8 cats. How many pets?

8 kg increased by 5 kg

## The Last 12 Facts (New)

*These facts include: 5 + 3, 6 + 3, 7 + 3 and their commutative pairs 3 + 5, 3 + 6, 3 + 7; 6 + 4, 7 + 4 and their commutative pairs 4 + 6, 4 + 7; 7 + 5 and 5 + 7.*

These facts can be done by a number of strategies. Present them to your students and encourage them to invent strategies to find the solutions. Some possible strategies with examples are presented below, but there are other viable strategies.

**a) 2-Apart Facts** *3 + 5, 4 + 6, 5 + 7 and 5 + 3, 6 + 4, 7 + 5*

If 1 from the larger of the two numbers is transferred to the smaller, a double is produced: this double is the number between the two given numbers. If students made Unifix towers for the two numbers, they could move one cube from the larger to the smaller and see the resultant double. Some students have trouble with this strategy because both numbers change so they are doubling a number that they don't actually see in front of them.

Another strategy for these facts involves removing 2 from the larger to make a double of the smaller, so the smaller is doubled and the 2 is added back on. Again, if students used Unifix cubes to model this strategy, they would see the double plus 2.

**Examples**

For 3 + 5, think: Transferring 1 from the 5 to the 3 makes 4 + 4, which is 8; or Double 3 is 6 and plus 2 is 8.

For 7 + 5, think: Transferring 1 from the 7 to the 5 makes 6 + 6, which is 12; or Double 5 is 10 plus 2 is 12.

**Examples of Some Practice Items**

4 + 6

$5 and $3

5¢ more than 7¢

**b) Plus Three Facts** *5 + 3. 6 + 3, 7 + 3 and 3 + 5, 3 + 6, 3 + 7*

A strategy for these facts is to add the 3 in two steps: first add 2 and then add 1.

**Examples**

For 6 + 3, think: 6 plus 2 is 8, and 8 plus 1 is 9.

For 3 + 7, think: 7 plus 2 is 9, and 9 plus 1 is 10.

**Examples of Some Practice Items**

$5 + $3

The sum of 3 and 6

7 kg increased by 3 kg

**c) Make-10 with a 7** *7 + 3, 7 + 4, 7 + 5 and 3 + 7, 4 + 7, 5 + 7*

A strategy is to visualize a ten frame for 7, note it would take 3 to make 10, and then add on any leftovers from the other number.

**Examples**

For 7 + 5, think: 7 plus 3 is 10, and 10 plus 2 (the other part of 5) is 12.

For 4 + 7, think: 7 plus 3 is 10, and 10 plus 1 (the other part of 4) is 11.

**Examples of Some Practice Items**

$3 + $7

7 red balloons and 5 white balloons. How many balloons altogether?

4 hours after 7 pm

# B. Addition — Mental Calculation

## Ten and “Some More” (Review)

From their experiences in grade 1, most students know that teen numbers can be made with a ten and “some more” even with limited knowledge of place value. They need to experience finding sums with a 10 and a single-digit number to be convinced they do not need to count on: the answer is automatic. This strategy should be reinforced before fact strategy learning is undertaken.

**Examples**

For 10 + 5, think: Ten and five makes 15 (fifteen).

For 7 + 10, think: Seven and ten makes 17 (seventeen).

**Examples of Some Practice Items**

10 + 3

8 girls and 10 boys. How many children?

5 small cubes and 1 rod

10 minutes and 9 minutes. What is the total time?

1 dime and 6 pennies

\$1 + \$10

## Tens and “Some More” (New)

As students develop understanding of place value, they realize that the answers to the addition of single-digit numbers to multiples of 10 (20, 30, 40, …, 90) are as easy as adding single digits to 10. Answers can be quickly stated without finger counting or counting on. Through place-mat activities and ten-frame activities in regular classroom time, students should become convinced of this easy addition. Afterwards, these addition questions can be reinforced in mental math time.

**Examples**

For 30 + 5, think: Thirty and five makes 35 (thirty-five).

For 7 + 80, think: Eighty and seven makes 87 (eighty-seven).

**Examples of Some Practice Items**

40 + 8

50 girls and 6 boys. How many children?

7 small cubes and 8 rods

9 minutes more than 30 minutes

5 dimes and 4 pennies

| | | |
|---|---|---|
| \$3 | 20 | 2 |
| + \$40 | + 5 | + 70 |

## Quick Addition (New)

Initially you should have students add 1-digit numbers to 2-digit numbers for questions that require no regrouping. Students need to be convinced that questions such as 32 + 7 and 74 + 5 are as easy as 2 + 7 and 4 + 5. Therefore, these questions should be modelled using ten frames so students can see that 32 (3 full ten frames and a 2) plus 7 just requires the addition of the 2 and 7, or that 74 (7 full ten frames and a 4) plus 5 just requires the addition of 4 and 5. The single digit facts required are those with sums less than 10: these facts were learned in grade 1 through ten frame visualization. Therefore, these questions could be reinforced before fact strategy learning in grade 2.

**Example**

For 34 + 3, think: 4 and 3 is 7, so the answer is 37 (thirty-seven).

For 81 + 8, think: 1 and 8 is 9, so the answer is 89 (eighty-nine).

**Examples of Some Practice Items**

43 + 3

52 girls and 6 boys. How many children?

$5 added to $72

35 cm + 2 cm

23¢ plus 5¢

| | | |
|---|---|---|
| $3 | 24 | 2 |
| + $42 | + 5 | + 76 |

Later in the year, after students have worked extensively with base-10 materials to model the addition of two 2-digit numbers (with and without regrouping) they can apply “quick addition” as an addition strategy for solving combinations that require no regrouping. You should select combinations that will result in answers under 100.

**Example**

For 56 + 23, think and record: 5 tens and 2 tens is 7 tens, and 6 and 3 is 9, so the answer is 79 (seventy-nine).

For 24 + 34, think and record: 2 tens and 3 tens is 5 tens, and 4 and 4 is 8, so the answer is 58 (fifty-eight).

**Examples of Some Practice Items**

56 + 31

24 maple trees and 24 elm trees. How many trees altogether?

$25 added to $72

One box is 45 kg and the other box is 32 kg. What is the total mass?

The 65-cm length is increased by 12 cm. What is the length now?

| | | |
|---|---|---|
| $13 | 24 | 22 |
| + $42 | + 35 | + 76 |

## Addition Facts to 10 Applied to Multiples of 10 (New)

Through modelling with small cubes and rods from the base-10 materials, students should be convinced that adding two sets of rods is no different than adding two sets of small cubes. For example, adding 4 rods and 2 rods results in 6 rods in the same way that adding 4 small cubes and 2 small cubes results in 6 small cubes. Therefore, when asked to find sums such as 20 + 30, 40 + 10, and 30 + 50, students should make the connections to the facts 2 + 3, 4 + 1 and 3 + 5.

You should restrict questions to combinations that result in sums to 100. Students would solve these questions by applying their knowledge of facts to 10. A response time goal of 5 seconds would be reasonable for these questions in grade 2.

**Example**

For 50 + 20, think: 5 tens and 2 tens is 7 tens, or 70 (seventy).

For 20 + 30, think: 2 tens and 3 tens is 5 tens, or 50 (fifty).

**Examples of Some Practice Items**

60 + 30

30 girls and 30 boys. How many children?

$20 added to $70

90 + 10

40 more than 50

80 plus 20

## Addition on the Hundreds Chart (New)

Display a Hundreds Chart, present students with addition questions involving two 2-digit numbers, ask them to visualize the additions on the chart, and record (or state) their answers. After many experiences, students may be able to visualize the Hundreds Chart and do the addition completely in their heads. In grade 3, this strategy will be the foundation for the *Break-Up-And-Bridge* strategy.

**Examples**

For 45 + 21, think: Starting at 45, go down two spaces to 65 and then go 1 space to the right to 66. The answer is 66.

For 34 + 63, think: Starting at 34, go down 6 spaces to 94 and then go 3 spaces to the right to get to 97; or Starting at 63, go down 3 spaces to 93 and then go 4 spaces to the right to get to 97. The answer is 97.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **10** | **11** | **12** | **13** | **14** | **15** | **16** | **17** | **18** | **19** |
| **20** | **21** | **22** | **23** | **24** | **25** | **26** | **27** | **28** | **29** |
| **30** | **31** | **32** | **33** | **34** | **35** | **36** | **37** | **38** | **39** |
| **40** | **41** | **42** | **43** | **44** | **45** | **46** | **47** | **48** | **49** |
| **50** | **51** | **52** | **53** | **54** | **55** | **56** | **57** | **58** | **59** |
| **60** | **61** | **62** | **63** | **64** | **65** | **66** | **67** | **68** | **69** |
| **70** | **71** | **72** | **73** | **74** | **75** | **76** | **77** | **78** | **79** |
| **80** | **81** | **82** | **83** | **84** | **85** | **86** | **87** | **88** | **89** |
| **90** | **91** | **92** | **93** | **94** | **95** | **96** | **97** | **98** | **99** |

**Examples of Some Practice Items**

$35 + $23

45 marigolds and 35 daisies. How many plants?

27 increased by 12

25 more than 72

29¢ and 32¢

What is the sum of 45 and 27?

23, 33, 43, __, ___, ___

## Finding Compatibles (New)

This strategy can be applied when students have to add three, or more, single-digit numbers. It involves looking for pairs of numbers that add to 10 to make the addition easier. Students first need to be convinced that they can add the numbers in any order without changing the answer (the associative property of addition).

**Examples**

For 3 + 8 + 7, think: First add the 3 and 7 to get 10, and then add the 8 to get 18.

For 8 + 1 + 6 + 2 + 9, think: 8 and 2 is 10, 9 and 1 is 10, 10 and 10 is twenty, and 20 plus 6 is 26.

**Examples of Some Practice Items**

2¢ + 3¢ + 8¢

$5 + $7 + $5

What is the sum of 9, 6, and 1?

8 + 1 + 9 + 2

3 cm + 9 cm + 7 cm

9 kg + 4 kg + 2 kg + 1 kg + 8 kg

# C. Subtraction — Fact Learning

## Subtraction Facts with Minuends of 10 or Less (Review)

In grade 1, through visualization of ten frames, students learned subtraction facts with minuends of 10 or less. This strategy involves visualizing the ten-frame dot configuration for a number, mentally removing the required number of dots, visualizing the resultant dot configuration, and naming this number. For some facts with minuends of 5 or 10, some students might just visualize the ten-frame configuration of the subtrahend, and know the difference is the number of empty cells in first row or the second row. A full top row that represents 5 and a full ten frame that represents 10 provide students with anchors for visualization of the numbers and operations.

**Examples**

For 5 – 2, think: For 5, I see a full top row. If 2 are dots are removed, I see the difference is 3; or think:
I see 2 dots in the top row and 3 empty cells, so the difference is 3.

For 7 – 3, think: For 7, I see the top row full and 2 in the bottom row.
If 2 dots are removed, I see 5, and if 1 more dot is removed, I see the difference is 4.

For 9 – 3, think: For 9, I see the top row full and 4 in the bottom row. If 3 dots are removed from the bottom row, I see a full top row and 1 in the bottom row, so I see the difference is 6.

For 10 – 7, think: For 10, I see a full ten frame. If 5 are dots are removed, I see 5 and if 2 more dots are removed I see the difference is 3; or think: I see 7 dots in ten frame and 3 empty cells, so the difference is 3.

**Examples of Some Practice Items**

8 – 2

$10 – $3

7 children. 5 girls. How many boys?

5 decreased by 4

What is the difference between 6 and 2?

7 minus 4

## Think Addition (New)

This strategy involves students finding answers to subtraction facts by using their knowledge of addition facts and their understanding of the inverse relationship of addition and subtraction. Instead of counting back to subtract, students ask themselves what they would have to add to the subtrahend to get the minuend. Obviously, this strategy will not generate quick responses unless students are competent with the addition facts. Therefore, you should not attempt to reinforce this strategy to get 3-second responses for the full set of subtraction facts unless all the addition facts have been mastered. You should start by applying this strategy to the subtraction facts with minuends of 10 or less, and to clusters of facts that are related to the better-known addition facts, such as the double facts and the plus-one facts. The full set of subtraction facts with an expected 3-second response has been assigned to grade 3 mental math.

**Examples**

For 10 – 5, think: What would I have to add to 5 to get 10? Since I know 5 + 5 = 10, the answer is 5.

For 9 – 8, think: What would I have to add to 8 to get 9? Since I know 8 + 1 = 9, the answer is 1.

**Examples of Some Practice Items**

10 – 3

What is 2 less than 8?

$14 – $7

What is the difference between 18 and 9?

What is left after 6 kg is reduced by 5 kg?

9 minus 1

# D. Subtraction — Mental Calculation

## Facts with Minuends of 10 or Less Applied to Multiples of 10 (New)

Through modelling with small cubes and rods from the base-10 materials, students should be convinced that subtracting two sets of rods is no different than subtracting two sets of small cubes. For example, subtracting 3 rods from 9 rods results in 6 rods in the same way that subtracting 3 small cubes from 9 small cubes results in 6 small cubes. Therefore, when asked to find differences such as 50 – 10, 40 – 20, and 90 – 50, students should make the connections to the facts 5 – 1, 4 – 2, and 9 – 5.

Questions should involve minuends of 20, 30, 40, …,100 so students can solve the questions by applying their knowledge of subtraction facts to 10. A response time goal of 5 seconds is reasonable for these questions at grade 2.

**Example**

For 50 – 20, think: 5 tens minus 2 tens is 3 tens, or 30 (thirty).

For 70 – 30, think: 7 tens subtract 3 tens is 4 tens, or 40 (forty).

**Examples of Some Practice Items**

60 + 30

30 girls and 30 boys. How many children?

$20 added to $70

90 + 10

40 more than 50

80 plus 20

## Quick Subtraction

This strategy is used when two 2- or 3-digit numbers are to be subtracted and there is no regrouping needed. Starting at the highest place value, simply subtract and record each place value’s digits. Because this strategy only applies to questions with no regrouping, students must holistically examine the demands of each question as a first step: this habit of thinking needs to pervade all mental math lessons. Students should be able to apply this strategy mentally to the differences between two 2-digit numbers and state the answer; however, most students will likely need to record each place-value difference for two 3-digit numbers and read their answers.

One suggestion for an activity during the discussion of this strategy is to present students with a list of 20 questions, some of which require regrouping, and direct students to apply quick subtraction to the appropriate questions, leaving out the ones for which this strategy cannot be used.

**Examples**

For 56 – 12, think about each place-value difference: Starting at the front end, 5 minus 1 is 4, 6 minus 2 is 4, so, the answer is 44 (forty-four).

For 57 – 41, think and record each place-value difference: Starting at the front end, 5 minus 4 is 1, 7 minus 1 is 6, so the answer is 16 (sixteen).

**Examples of Some Practice Items**

Examples of Some Practice Items for Numbers in the 10s:

76 – 25

The difference between 18 and 39

$97 – $22

A mass of 46 kg reduced by 15 kg

35 children. 14 girls. How many boys?

69 eggs. 1 dozen removed. How many eggs left?

## Subtraction on the Hundreds Chart (New)

This strategy involves actions on a Hundreds Chart: Going up where each space represents subtracting 10, and going left where each space represents subtracting 1. Display a Hundreds Chart, present students with subtraction questions involving two 2-digit numbers, ask them to visualize the subtractions on the chart, and record (or state) their answers. After many experiences, students may be able to visualize the Hundreds Chart and do the subtraction completely in their heads.

**Examples**

For 45 – 21, think: Starting at 45, go up two spaces to 25 and then go 1 space to the left to 24. The answer is 24.

For 74 - 65, think: Starting at 74, go up 6 spaces to 14 and then go 5 spaces to the left to get to 9. The answer is 9

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **10** | **11** | **12** | **13** | **14** | **15** | **16** | **17** | **18** | **19** |
| **20** | **21** | **22** | **23** | **24** | **25** | **26** | **27** | **28** | **29** |
| **30** | **31** | **32** | **33** | **34** | **35** | **36** | **37** | **38** | **39** |
| **40** | **41** | **42** | **43** | **44** | **45** | **46** | **47** | **48** | **49** |
| **50** | **51** | **52** | **53** | **54** | **55** | **56** | **57** | **58** | **59** |
| **60** | **61** | **62** | **63** | **64** | **65** | **66** | **67** | **68** | **69** |
| **70** | **71** | **72** | **73** | **74** | **75** | **76** | **77** | **78** | **79** |
| **80** | **81** | **82** | **83** | **84** | **85** | **86** | **87** | **88** | **89** |
| **90** | **91** | **92** | **93** | **94** | **95** | **96** | **97** | **98** | **99** |

**Examples of Some Practice Items**

56 – 32

$47 – $23

67 decreased by 22

What is the difference between 78 kg and 19 kg?

How much more is $85 than $34?

33¢ subtract 15¢

# E. Addition and Subtraction — Computational Estimation

## Front-End Estimation (New)

Grade 2 students should use this strategy to estimate sums and differences of two 2-digit numbers before they get exact answers by pencil-and-paper strategies or by technology. This strategy involves combining only the values in the highest place value to get a "ball-park figure". In real life situations, such front-end estimates are often all that is needed.

**Examples**

To estimate 32 + 45, think: 3 tens and 4 tens is 7 tens, or 70, so the estimate is more than seventy.

To estimate 76 – 34, think: 7 tens minus 3 tens is 4 tens, or 40, so the estimate is about forty.

**Examples of Some Practice Items**

Estimate: 45
+ 36

Estimate the sum of $56 and $23

Approximately what is 45 kg reduced by 17 kg?

About what is the difference between 67 and 25?

Approximately what is 36 more than 62?

Estimate what you save if you buy a game for $52 rather than $76.

## Rounding (New)

Grade 2 students should use this strategy to estimate sums and differences of two 2-digit numbers when closer estimates are needed than the estimates provided by the front-end strategy. These estimates are best in situations where only an estimate is needed. This strategy involves rounding each number to the nearest tens and adding these tens.

When the ones digit is 5, students can opt to round up or down. If two numbers in an addition question have 5s as ones digits, round one of the numbers up and one of them down. In subtraction, both numbers with 5s in the ones digits should be rounded up or down. (For two 2-digit numbers with 5s in the ones place, this estimation strategy actually produces exact answers!)

**Examples**

To estimate 27 + 31, think: Round 27 to 30 and round 31 to 30 to get 30 plus 30, which is an estimate of 60 (sixty).

To estimate 25 + 45, think: Both numbers have 5s so round 25 to 30 and 45 to 40 to get 30 plus 40, which is an estimate of 70 (seventy).

To estimate 81 – 32, think: Round 81 to 80 and 32 to 30 to get 80 minus 30, which is an estimate of 50 (fifty).

To estimate 52 – 29, think: Round 52 to 50 and 29 to 30 to get 50 minus 30, which is an estimate of 20 (twenty).

To estimate 65 – 25, think: Both numbers have 5s so round 65 to 70 and 25 to 30 to get 70 – 30, which is an estimate of 40 (forty).

**Examples of Some Practice Items**

Estimate: \$34 + \$56

Approximately what is the sum of 28 kg and 48 kg?

About how many bottles were collected if one group had 44 bottles and the other group had 55 bottles?

Estimate: 82 – 33

Approximately what is the difference between \$65 and \$35.

About how much more is the 82¢ chocolate bar than the 59¢ one?

# PART 2

## Measurement Estimation

# The Implementation of Measurement Estimation

## General Approach

A measurement estimation strategy would normally be reinforced and assessed during mental math time in the grades following its initial introduction. The goal in mental math is to increase a student's *competency* with the strategy. It is expected that measurement estimation strategies would be introduced as part of the general development of measurement concepts at the appropriate grade levels. Each strategy would be explored, modeled, and reinforced through a variety of class activities that promote understanding, the use of estimation language, and the refinement of the strategy. As well, the assessment of the strategy during the instructional and practice stages would be ongoing and varied.

*Competency* means that a student reaches a reasonable estimate using an appropriate strategy in a reasonable time frame. While what is considered a reasonable estimate will vary with the size of the unit and the size of the object, a general rule would be to aim for an estimate that is within 10% of the actual measure. A reasonable time frame will vary with the strategy being used; for example, using the *benchmark strategy* to get an estimate in metres might take 5 to 10 seconds, while using the *chunking strategy* might take 10 to 30 seconds, depending upon the complexity of the task.

## Introducing a Strategy

Measurement estimation strategies should be introduced when appropriate as part of a rich and carefully sequenced development of measurement concepts. For example, in grade 3, the distance from the floor to most door handles is employed as a *benchmark* for a metre so students can use a *benchmark strategy* to estimate lengths of objects in metres. This has followed many other experiences with linear measurement in earlier grades: in grade primary, students compared and ordered lengths of objects concretely and visually; in grade 1, students estimated lengths of objects using non-standard units such as paper clips; in grade 2, students developed the standard unit of a metre and were merely introduced to the idea of a *benchmark* for a metre.

The introduction of a measurement estimation strategy should include a variety of activities that use concrete materials, visuals, and meaningful contexts, in order to have students thoroughly understand the strategy. For example, a meaningful context for the *chunking strategy* might be to estimate the area available for bookshelves in the classroom. Explicit modeling of the mental processes used to carry out a strategy should also be part of this introduction. For example, when demonstrating the *subdivision strategy* to get an estimate for the area of a wall, you would orally describe the process of dividing the wall into fractional parts, the process of determining the area of one part in square metres, and the multiplication process to get the estimate of the area of the entire wall in square metres. During this introductory phase, you should also lead discussions of situations for which a particular strategy is most appropriate and efficient, and of situations for which it would not be appropriate nor efficient.

## Reinforcement

Each strategy for building measurement estimation skills should be practised in isolation until students can give reasonable estimates in reasonable time frames. Students must understand the logic of the strategy, recognize when it is appropriate, and be able to explain the strategy. The amount of time spent on each strategy should be determined by the students' abilities, progress, and previous experiences.

The reinforcement activities for a strategy should be varied in type and should focus as much on the discussion of how students obtained their answers, as on the answers themselves. The reinforcement activities should be structured to ensure maximum participation. At first, time frames should be generous and then narrowed as students internalize the strategy and become more efficient. Student participation should be monitored and their progress assessed in a variety of ways. This will help determine the length of time that should be spent on a strategy.

During the reinforcement activities, the actual measures should not be determined every time an estimate is made. You do not want your students to think that an estimate is always followed by measurement with an instrument: there are many instances where an estimate is all that is required. When students are first introduced to an activity, it is helpful to follow their first few estimates with a determination of the actual measurement in order to help them refine their estimation abilities. Afterwards, however, you should just confirm the reasonable estimates, having determined them in advance.

Most of the reinforcement activities in measurement will require the availability of many objects and materials because students will be using some objects and materials as benchmarks and will be estimating the attributes of others. To do this, they must see and/or touch those objects and materials.

After you are confident that most students have achieved a reasonable competency with the strategy, you need to help them integrate it with other strategies that have been developed. You should do this by providing activities that include a mix of attributes to be estimated in different units using a variety of contexts. You should have the students complete the activities and discuss the strategy/strategies that were, or could have been, used. You should also have students match measurement estimation tasks to a list of strategies, and have them discuss the reasoning for their matches.

## Assessment

Your assessments of measurement estimation strategies should take a variety of forms. Assessment opportunities include making and noting observations during the reinforcements, as well as students' oral and written responses and explanations. Individual interviews can provide you with many insights into a student's thinking about measurement tasks. As well, traditional quizzes that involve students recording answers to estimation questions that you give one-at-a-time in a certain time frame can be used.

Assessments, regardless of their form, should shed light on students' abilities to estimate measurements efficiently and accurately, to select appropriate strategies, and to explain their thinking.

## Measurement Estimation Strategies

**The Benchmark Strategy**

This strategy involves mentally matching the attribute of the object being estimated to a known measurement. For example, to estimate the width of a large white board, students might mentally compare its width to the distance from the doorknob to the floor. This distance that is known to be about 1 metre is a *benchmark.* When students mentally match the width of the white board to this benchmark, they may estimate that the width would be about two of these benchmarks; therefore, their estimate would be 2 metres. In mathematics education literature you will often see reference made to *personal referents.* These are benchmarks that individuals establish using their own bodies; for example, the width of a little finger might be a personal referent for 1 cm, a hand span a referent for 20 cm, and a hand width a referent for 1 dm. These benchmarks have the advantage of being portable and always present whenever and wherever an estimate is needed.

**The Chunking Strategy (Starting in Grade 5)**

This strategy involves mentally dividing the attribute of an object into a number of chunks, estimating each of these chunks, and adding the estimates of all the chunks to get the total estimate. For example, to estimate the length of a wall, a student might estimate the distance from one corner to the door, estimate the width of the door, estimate the distance from the door to the bookcase, and estimate the length of the bookcase which extends to the other corner. All these individual estimates would be added to get the estimate of the length of the wall.

**The Unitizing Strategy (Starting in Grade 5)**

This strategy involves establishing an object (that is smaller than the object being estimated) as a unit, estimating the size of the attribute of this unit, and multiplying by the number of these units it would take to make the object being measured. For example, students might get an estimate for the length of a wall by noticing it is about five bulletin boards long. The bulletin board becomes the unit. Students estimate the length of the bulletin board and multiply this estimate by five to get an estimate of the length of the wall.

**The Subdivision Strategy (Starting in Grade 6)**
This strategy involves mentally dividing an object repeatedly in half until a more manageable part of the object can be estimated and then this estimate is multiplied by the appropriate factor to get an estimate for the whole object. For example, to estimate the area of a table top, students might mentally divide the table top into fourths, estimate the area of that one-fourth in square decimeters, and then multiply that estimate by four to get the estimate of the area of the entire table top.

# F: Measurement Estimation

In grade 1, students developed strategies for estimating length, capacity, mass, and area in non-standard units, as part of the general development of these measurement concepts. In grade 2, in mental math time, students should become more competent in estimating with non-standard units by getting reasonable estimates in reasonable time frames.

You are encouraged to use a variety of response formats to try to get maximum participation. Many students are initially unwilling to commit themselves to a single number estimate, or they give “wild guesses.” If, however, they can select from a list of ranges of possible estimates, or can respond with a comparison to a given amount (less than, more than, or about), or can select a single-number estimate in a multiple-choice format, they are more likely to participate. Then through discussions of responses and strategies, and more experiences, they will refine their abilities to get reasonable estimates. Eventually they will reach the point where they can generate their own single-number estimates.

Although students in grade 2 are working with numbers to 999, in measurement estimation questions you should restrict the size of the numbers involved to those students can easily visualize.

## Measurement Estimation — Length

Estimate and compare lengths using non-standard units such as hand spans, foot strides, finger widths, and eraser lengths.

**Examples of Some Practice Activities**

Gather five objects that you will individually show to students and ask them if its length is less than, greater than, or about the same as the length of a sheet of paper. You could monitor responses by asking students to raise their left hand if they think it is less, their right hand if they think it is more, and no hand if they think its is about the same. After the estimation for each object, do a direct comparison of that object to a piece of paper.

Gather five objects that would be appropriate to compare to finger widths, such as an eraser, crayon box, paper clip, file card, and CD. Ask students to choose between two options that you provide. For example, ask, Is the length of this eraser closer to 4 or to 7 finger widths? To monitor responses, you could opt to write a 4 and a 7 as headings on the whiteboard and have students place a check mark or a post-it under their choice. Discuss how they made their decisions. For first two, do a direct comparison to finger widths to check which would be the better estimate, and encourage students to use these checks to help them refine their estimating abilities.

Prepare a handout of a list of objects in the classroom in one column and a blank column beside it. Provide each student with the list and explain that you will ask them to estimate how many of their hand spans would be need to measure the length, width, or height of each object. They should record their estimates on their handout in the column opposite the name of each object. Start at the top of the list and, one-at-a-time, ask students to estimate number of hand spans for either the length or the width of the object. For example, depending upon what objects you put in the list, you might ask for the width of your desk, the length of the bookshelf, or the height of the doorknob. Afterwards, students could move around the room in partnerships and check their estimates with their hand spans.

On the whiteboard provide the students with three ranges: 1 to 3, 4 to 6, and 7 to 9. Explain that you are going to ask them to estimate how far apart are two objects in the classroom. They are to select one of the three ranges of numbers of foot strides that are on the board. After each of the first two or three estimates, ask students to share how they made their decisions and then determine the actual numbers to help students refine their estimation abilities. For the other estimates, especially if the majority of students are correct, you just indicate the best range and get one or two students who had the correct range to share how they made their decisions.

Provide each student with a paper clip. Select ten objects that will have a length between 1 and 20 paper clips. One-at-a-time, display the objects and ask students to estimate the number of paper clips long each one is. Explain that they are to write one number between 1 and 20 on their individual white boards. Any estimate that is within 2 paper clips of the actual length is a very good estimate; for example, if the actual length of a book is 12 paper clips, any estimate from 10 to 14 would be a very good estimate. After each of the first three or four objects have been estimated, have students explain how they made their decisions, measure the length with paper clips to get the actual number, and help students refine their estimation abilities. For the others, lead a discussion of how estimates were made and just indicate which numbers would be very good estimates.

## Measurement Estimation — Capacity

Estimate and compare capacities using non-standard units such as cupfuls of sand, water, or rice; Unifix cubes; marbles; or Styrofoam packing peanuts.

**Examples of Some Practice Activities**

Assemble a number of objects that have capacity from 1 to 20 cups of water, such as a small bowl, different sizes of paper cups, plastic container, large juice can, etc. Show the students a Styrofoam cup filled with water. Explain that you are going to show them other containers and they are to estimate if each container holds 1 – 5, 6 – 10, 11 – 15, or 16 – 20 Styrofoam cups of water. Write these intervals as headings on the white board. After students have estimated the capacity of each object, they can quickly place a post-it under the heading of their choice. Discuss their strategies and check the first few by pouring and counting the number of cups of water used. Help students refine their estimation abilities.

Gather a number of containers that would hold from 25 to 100 Styrofoam packing peanuts, such as bowls, plastic containers, small garbage can, large milk carton, shoe box, etc. Have students select an estimate of the number of peanuts that would fill each container from a list of four choices that you provide. For example, you would display a plastic container that you previously determined held 65 peanuts, and ask your students to choose which of the following is the best estimate for the number of peanuts it contains: 40, 60, 80, 100.

Fill a number of small medicine dispensing cups with rice. Select a few containers that would be filled by 1 to 20 of these cups. Display the cups of rice and one of the containers that you previously determined was filled by 14 cups of rice. Ask the students to decide if the container would be filled by less than 10 cups, about 10 cups, or more than 10 cups. Put the three choices on the white board and ask for a show of hands for each choice, recording the number of students who selected each one. Discuss how they made their decisions. Fill the container with cups of rice as the class counts to keep track. Repeat this for a few more containers of different capacities. (Be sure to give the students choices that are reasonable for the container.) Remind students that the capacities of the containers they have already measured can help them make decisions for others.

Assemble a collection of tin cans of various sizes and shapes with their labels and tops off. Fill one of the cans with marbles. Ask students to estimate how many marbles they think are in the can. Discuss their estimates and strategies used to make the estimate. Count the marbles to get the actual number and record it on the white board. As you show each of the other cans, one at a time, ask students to use the known quantity to help them estimate the number of marbles it would hold. Have them record each estimate on their individual white boards and show you when called on. Discuss how they made their estimates. Check each one by filling and counting. Indicate the range of numbers of marbles that would be very good estimates (use 10 – 15%). After each can, note the spreads in estimates on the white boards; hopefully, you should see the estimates improve with experience. This activity could be repeated using a different material as the fill.

## Measurement Estimation — Mass

Estimate and compare masses in non-standard units such as Unifix cubes, wooden blocks, paper cups of sand and books. This is the most difficult measurement to estimate because it uses the sense of touch rather than sight, and in fact sight can be a hindrance for students who associate larger objects with larger mass. For example, if you had 1 kilogram of bricks and 1 kilogram of feathers, the feathers would "look" so much larger. Students would have to have had many experiences lifting and comparing objects before they would be able to make good judgements about mass without actually lifting the objects.

**Examples of Some Practice Activities**

Provide each student with 2 or 3 trapezoids from a set of Pattern Blocks. Think of a number of objects that each student would have readily available and that would have masses of 20, or less, trapezoids. Ask them to estimate how many trapezoids would have the same mass as these objects. For example, you might use a calculator, a ruler, a scribbler, and bottle of glue. After you name each object they are to estimate, have them write their estimates on the individual whiteboards to display when you ask. Discuss how they made their decisions. Have a pan balance ready to check the actual number of trapezoids. (Initially, you could provide students with options, such as intervals or a list of numbers.)

Prepare ten numbered stations, each of which has an object that is less than, more than, or about the same mass as 1 textbook. Provide students with a handout that has a column with the station number, and three columns (less than, about the same, more than). Ask students to take their recording sheet and a textbook, visit each station to make an estimate, and record their decision using a checkmark in the appropriate column opposite the station number. After all students have completed their visits, discuss each one after you have had a show of hands.

For each group of 4 students, tape shut a small paper bag containing 10 hexagons from a set of Pattern Blocks. This will be the unit of mass for the activity. Select a few objects that you predetermined have masses of 10 or less of these bags. Provide each group with a set of these objects and ask them to estimate how many bags would have the same mass as each of the objects. Record the group estimates for each object on the whiteboard. Discuss how they made their decisions, and provide them with the range of good estimates.

Have each group of four students gather five objects of their choice, or ones that you choose. (Each group could have a different set.) Select an appropriate unit for each group to use depending upon the objects they gathered. Ask them to order their objects from smallest to largest mass. Have them use a pan balance to see how many of their assigned units are needed to balance the object. Using the number they obtained for the smallest, they should estimate the number of units for the other objects. They could check their estimates by continuing to use the pan balance.

## Measurement Estimation — Area

Estimate and compare areas using non-standard units such as post-it notes, CD covers, sheets of paper or newspaper, pattern blocks, and index cards

**Examples of Some Practice Activities**

Provide each student with a post-it of one size. Explain that this is the unit of area that they will be using for today's activity. Ask, Will the area of your textbook cover be between 18 and 24 post-its or between 30 and 36 post-its? (Note: You will change these choices depending upon the size of the post-its.) When called on, students should raise their left hand if they think 18 – 24 or their right hand if they think 30 – 36. Discuss how they made their decisions. Cover one textbook with post-its to get the actual number. Repeat using other surfaces and choices that are reasonable for those surfaces.

Put on display one sheet of newspaper. Explain that you will ask students to estimate how many of these sheets of newspaper it would take to cover a certain surface in the classroom. They will decide if their estimate is less than, more than or about a prescribed number. For example, ask students if the door of the classroom would be covered by less than 8 sheets, more than 8 sheets, or about 8 sheets. Write these three choices on the whiteboard and have students quickly place their post-its under their choice. Discuss how they made their decisions. Tape a few sheets of newspaper on the door until students are convinced of the best estimate. Repeat for other surfaces in the classroom, such as the table top, the whiteboard, or the reading area.

Prepare an outline of a shape that would be covered by 12 hexagons from s set of Pattern Blocks. Tape it on the whiteboard. Ask students to estimate how many hexagons it would take to cover the shape. They can choose from this set of numbers: 5, 8, 11, 15. Have them write their choice on their individual whiteboards and display the whiteboards when you ask them. Discuss how they made their decisions. Cover the shape with 5 hexagons and ask the students if they would like to change their minds based on seeing 5 in place. Continue placing hexagons on until students are convinced that 11 is the most reasonable estimate. Repeat using outlines of other shapes that are covered by a different number of hexagons, or use other Pattern Blocks.

Provide each student with a file card that will be the unit of area. Ask them to estimate how many file cards would cover their textbook cover and write the number on their individual whiteboards to display when called upon. Note the variety of responses and ask students to share how they made their decisions. Cover one textbook with file cards as students count them, making decisions about how many would be needed for the leftover space. Discuss that a good estimate would be between 3, 4, or 5. (Note: These numbers depend on the size of the file cards you use.) Repeat by asking students to estimate the area of other surfaces in file cards.

# PART 3

Spatial Sense

# The Development of Spatial Sense

## What is spatial sense?

*Spatial sense* is an intuition about shapes and their relationships, and an ability to manipulate shapes in one's mind. It includes being comfortable with geometric descriptions of shapes and positions.

There are seven abilities in spatial sense that need to be addressed in the classroom:

**Eye-motor co-ordination**. This is the ability to co-ordinate vision with body movement. In the mathematics classroom, this involves children learning to write and draw using pencils, crayons, and markers; to cut with scissors; and to handle materials.

**Visual memory**. This is the ability to recall objects no longer in view.

**Position-in-space perception**. This is the ability to determine the relationship between one object and another, and an object's relationship to the observer. This includes the development of associated language (over, under, beside, on top of, right, left, etc.) and the transformations (translations, reflections, and rotations) that change an object's position.

**Visual discrimination**. This is the ability to identify the similarities and differences between, or among, objects.

**Figure-ground perception.** This is the ability to focus on a specific object within a picture or within a group of objects, while treating the rest of the picture or the objects as background.

**Perceptual constancy**. This is the ability to recognize a shape when it is seen from a different viewpoint, or from a different distance. This is the perception at play when students recognize similar shapes (enlargements/reductions), and when they perceive as squares and rectangles, the rhombi and parallelograms in isometric drawings.

**Perception of spatial relationships.** This is the ability to see the relationship between/among two or more objects. This perception is central when students assemble materials to create an object or when they solve puzzles, such as tangram, pattern block, and jigsaw puzzles.

These abilities develop naturally through many experiences in life; therefore, most students come to school with varying degrees of ability in each of these seven aspects. However, students who have been identified as having a learning disability are likely to have deficits in one or more of these abilities. Classroom activities should be planned to allow most students to further develop these perceptions. Most spatial sense activities involve many, if not most, of the spatial abilities, although often one of them is highlighted.

## Classroom Context

Spatial abilities are essential to the development of geometric concepts, and the development of geometric concepts provides the opportunity for further development of spatial abilities. This mutually supportive development can be achieved through consistent and ongoing strategic planning of rich experiences with shapes and spatial relationships. Classroom experiences should also include activities specifically designed for the development of spatial abilities. These activities should focus on shapes new to the grade, as well as shapes from previous grades. As the shapes become more complex, students' spatial senses should be further developed through activities aimed at the discovery of relationships between and among properties of these shapes. Ultimately, students should be able to visualize shapes and their various transformations, as well as sub-divisions and composites of these shapes.

## Spatial Sense in Mental Math Time

Following the exploration and development of spatial abilities during geometry instruction, mental math time can provide the opportunity to revisit and reinforce spatial abilities periodically throughout the school year.

## Assessment

Assessment of spatial sense development should take a variety of forms. The focus in this aspect of mental math is on individual growth and development in spatial sense, rather than on an arbitrary level of competency to be achieved. You should record any observations of growth students make during the reinforcements, as well as noting students' oral and written responses and explanations. For spatial sense, traditional quizzes that involve students recording answers to questions that you give one-at-a-time in a certain time frame should play a very minor role.

# G: Spatial Sense

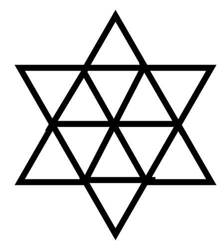
## Spatial Sense in 2-D Geometry

In grade 2, students recognize, name, describe, and represent parallelograms and continue to work with squares, triangles, circles, rectangles, rhombi, trapezoids and hexagons. They use the term right angle to describe a “square corner” in shapes. They subdivide and change more complicated constructions of 2-D figures. They recognize, identify, describe, and represent, reflective symmetry in 2-D shapes. They realize that the line of symmetry can be vertical, horizontal or slanted. They recognize parallel lines in real-world objects and represent parallel lines/segments using materials.

**Examples of Spatial Sense Activities**

Show a right-angle triangle on an overhead Geoboard. Ask students to replicate your shape on their individual Geoboards. After a reasonable time ask students to show you their Geoboards to check that they have the same right-angle triangle in the same position on their Geoboards. Repeat this activity, using larger/smaller right-angle triangles in other positions.

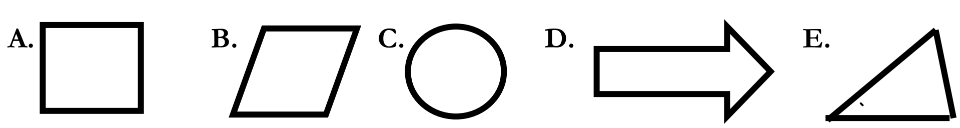
This activity could also be modified by asking students to concentrate on your overhead Geoboard shape for 8 – 10 seconds, after which time you shut off the overhead, and students try to replicate your shape from memory. Ask students to share how the committed your shape to memory.



Show students a picture, such as a star made of 12 pattern block triangles, and ask them how many triangles they can find.  After a reasonable time, ask them to record on their whiteboards the number of triangles they see. Record the answers you see, and lead a discussion of where the various triangles, particularly those that are overlapping. Repeat this activity, using other composite drawings of rectangles, squares, circles, rhombi, trapezoids and hexagons.

Draw the Described Shape. Ask students to make the shapes you describe on their whiteboards. After a reasonable amount of time, ask then to show their shapes. Some possible descriptions are: a) I have four equal sides and four right angles. b) I have two pairs of parallel sides, my opposite sides are equal and I have no right angles. c) I have one pair of parallel sides and no right angles.

Display pictures of five shapes, labeled A to E. Three of the shapes should have at least one line of symmetry and two of the shapes should have no lines of symmetry. Ask students to identify the symmetrical shapes by recording their letters on their individual whiteboards. Afterwards, ask individual students to locate the lines of symmetry.



## Spatial Sense in 3-D Geometry

In grade 2, students recognize, name, describe, and represent a rectangular pyramid and continue to use square prisms, triangular prisms, rectangular prisms, triangular pyramids and square pyramids. They name triangular, square, and rectangular prisms and pyramids. They describe 3-D shapes such as cylinders, spheres, and cones by their surfaces, both the curved surfaces and the faces (flat surfaces).

**Examples of Spatial Sense Activities**

What am I? Display a variety of 3-D shapes so students can see them. List the names of 3-D shapes that students should know and recognize. Explain that you will describe a shape and they have to select the 3-D shape that matches the description you give. They are to write the name of the shape on their individual whiteboards and show it when you direct them to do so. Some possible descriptions are:

– I have six faces. All my faces are squares. (Answer: cube)

– I have four faces. All my faces are triangles. (Answer: triangular prism)

– I have five faces. I have one square face. My other faces are triangles. (Answer: square pyramid)

Display a variety of 3-D shapes so students can see them. Place a picture of a net for one of the shapes on the overhead. Ask to select the 3-D shape that would be made with this net and record its name on their whiteboards. Ask them to show their answer and discuss how they made their decisions. Repeat by showing nets of other shapes.

Ask students to visualize a triangular prism. Explain that someone used this 3-D shape to make shapes in the sand on the beach by pressing its faces into the sand. What shapes could have been made in the sand? Ask them to write their answers on their whiteboards. Repeat by selecting other 3-D shapes.

For each group of students, place four 3-D shapes individually in four paper bags labelled A to D. When directed students should place their hands in the bags and, without looking, touch the shapes and try to determine which 3-D shape it is. On a piece of paper they should record the bag letter and the name. Have them exchange bags and repeat the process until every student has had each of the four bags.