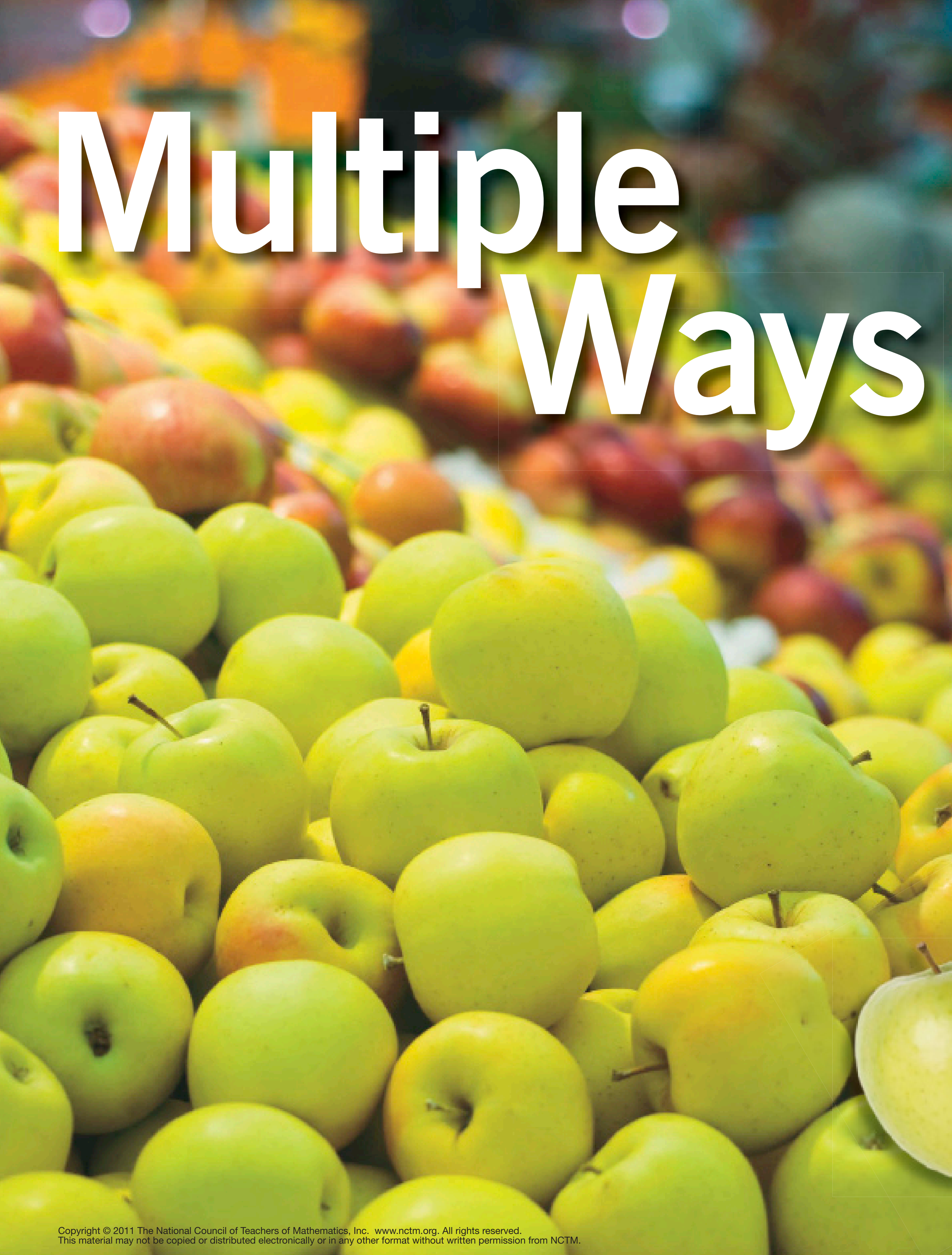
Multiple Ways



to Solve Proportions

Explore both intuitive approaches and standard algorithms to solve proportional reasoning problems.

Leslie K. Ercole, Marny Frantz, and George Ashline

When solving problems involving proportions, students may intuitively draw on strategies that connect to their understanding of fractions, decimals, and percents. These two statements—"Instruction in solving proportions should include methods that have a strong intuitive basis" (NCTM 2000, p. 221) and "Teachers should begin instruction with more intuitive strategies, such as the unit-rate and factor-of-change [that] emphasize learning concepts over learning procedures" (Cramer and Post 1993, http://www.cehd.umn.edu/rationalnumberproject/93_2.html)—will be explored in light of the various methods that students have at their disposal.

In addition to unit rate and factor of change, students can also transition to the equivalent fraction method and the building-up strategy. Although these methods are intuitive, they are inefficient and cumbersome when solving noninteger proportion problems. More structured methods are also available, such as using a ratio table, a graph, the standard algorithm, and cross multiplication.

THE UNIT-RATE METHOD

The unit-rate, or "How many for one?" strategy is an intuitive way to solve proportion problems. It is similar to how a student learned about unit fractions. "The unit-rate method is far and away the most widely used by children

Fig. 1 In this missing-value problem, the unit rate is the same as the unit cost.

The Healthy Food Store

The Healthy Food Store sells granola by the ounce. The cost depends on the weight of the granola. Granola that weighs 8 ounces costs $1.50. Fill in the table with the appropriate cost or weight. Explain your reasoning.

(a)

Weight (in Ounces)	Cost (in Dollars)
6	$1.13
8	$1.50
16	$3.00
	4.50

.1875

$1.50 ÷ 8 = .1875 – first step

6 × .1875 = $1.125 (1.13)

16 × .1875 = $3.00

$4.50 ÷ .1875 = 24

(b)

who have not had formal instruction in the standard cross-multiplication algorithm" (Post, Behr, and Lesh 1988, http://www.cehd.umn.edu/rationalnumberproject/88_10.html). To determine a unit rate, students must first recognize that a relationship exists between the two given quantities and then calculate the rate when one of the quantities is 1. See, for example, the Apple problem:

> A grocery store offers 6 apples [the quantity] for $1.50 [the quantity]. How much would 1 apple cost?

The rate of $1.50/6 apples is equivalent to $0.25/1 apple, which is the unit rate.

Once the unit rate has been calculated, the student can solve a missing-value problem by multiplying the quantity given in the problem by the unit rate. For example:

> A store sells 6 apples for $1.50. What is the cost of 20 apples?

The solution is 20 apples × $0.25 per apple (the unit rate) = $5.00.

It is important to note that every rate actually has two unit rates, which are known as *dual* or *reciprocal rates.* The Apple problem could have read:

> A grocery store sells 6 apples [quantity] for $1.50 [quantity]. How many apples can I buy with $1.00?

In the solution, 6 apples/$1.50 is equivalent to 4 apples/$1. The two unit rates in this example are *dollars per 1 apple* and *apples per 1 dollar*. Note that one of the two unit rates in a proportion is usually more understandable and easier to access for some students. Students should become aware that all rates have a dual or reciprocal rate.

The Healthy Food Store problem in **figure 1a** is a missing-value problem. In **figure 1b**, the student calculates a unit rate (in this case, a unit cost) by dividing $1.50 by 8 ounces of granola to equal $0.1875 per ounce of granola, then correctly uses the cost for 1 ounce of granola (the unit rate) to calculate the additional quantities needed in the table.

The Big Horn Ranch problem in **figure 2a** is a ratio-comparison problem. The student (see **fig. 2b**) uses a unit-rate approach and sets up each ranch as a ratio of acres to horses with clearly defined quantities. The student calculates a unit rate of acres of pasture per one horse, then compares the two unit rates to solve the problem correctly.

Figure 2c illustrates the confusion that students encounter when working with dual rates. After setting up ratios of acres of pasture (p) and horses (h), the student determines how many horses will be on 1 acre of pasture, or a rate of *horses per 1 acre*, not *acres per horse,* as the problem asks. The student then selects Big Horn Ranch as the solution, because the fraction 2/3 is greater than 3/5. This student does not recognize that the larger fraction represents more horses per acre of pasture. The student appears to misunderstand the unit rate

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of horses per acre, perhaps because it is less familiar than acres per horse.

FACTOR-OF-CHANGE METHOD

Another intuitive strategy that students use when solving a proportion is the factor-of-change method. It is also called the scalar method or size-change method. With similarity problems, it is referred to as the scale-factor method. This approach requires students to use "times as many" thinking. It focuses on the multiplicative relationships *within* and *between* the ratios in the problem. The missing-value Apple problem can be viewed in light of the factor of change:

$$\frac{24 \text{ apples}}{\$6} = \frac{x \text{ apples}}{\$18}$$

A student might look at the relationship *within* the ratio and determine that there are four times as many apples as dollars.

Note that when the student determines the factor of change within the ratio, he or she is also determining the unit rate, which is

$$\$6 \times \frac{4 \text{ apples}}{\$1} = 24 \text{ apples}.$$

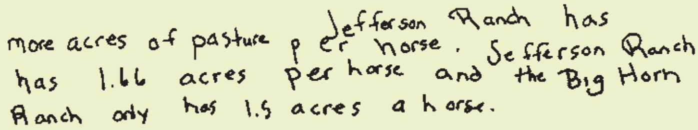
Next, this student would multiply the \$18 by 4 apples per dollar to obtain 72 apples.

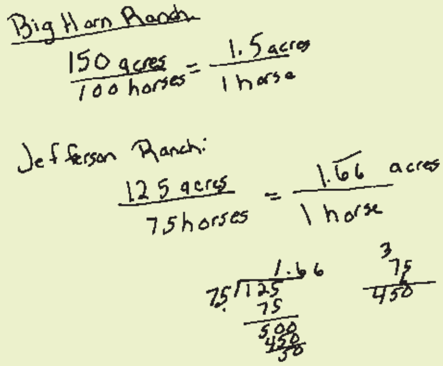
An alternate method involves the multiplicative relationship *between* the two ratios: There are three times as many dollars ($\$6 \times 3 = \$18$), so there are three times as many apples, or 24 apples $\times$ 3 = 72 apples.

Fig. 2 This example of a ratio-comparison problem looks at acres in relation to horses.

Big Horn Ranch Problem
Big Horn Ranch has 150 acres of pasture and raises 100 horses. Jefferson Ranch has 125 acres of pasture and raises 75 horses. Which ranch has more acres of pasture per horse? Explain your answer using words, pictures, or diagrams.

(a)





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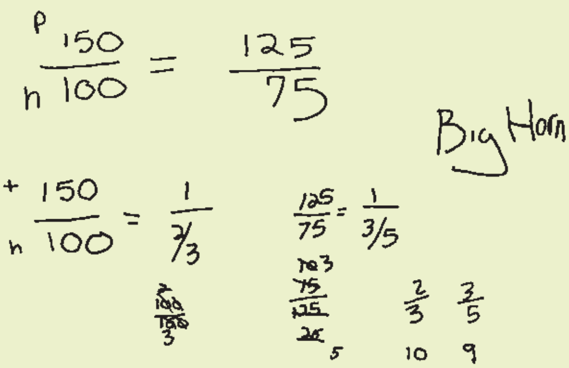
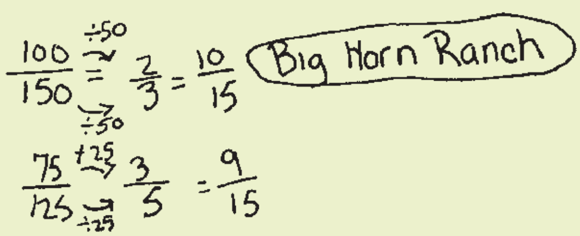


(c)



(d)

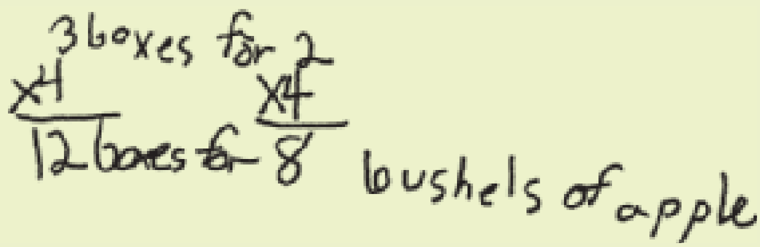
Sources: Student work samples from the Ongoing Assessment Project (OGAP) (Petit, Laird, and Marsden 2010) and Vermont Mathematics Partnership (VMP) Pilot, 2006–2007

Fig. 3 The factor of change between the two ratios is demonstrated with apples.

Apple Packing

Carrie is packing apples for an orchard's mail order business. It takes 3 boxes to pack 2 bushels of apples. How many boxes will she need to pack 8 bushels of apples?

(a)



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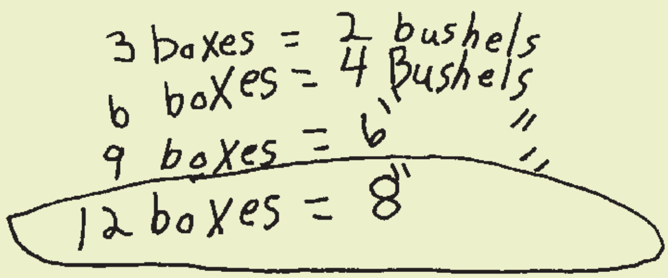


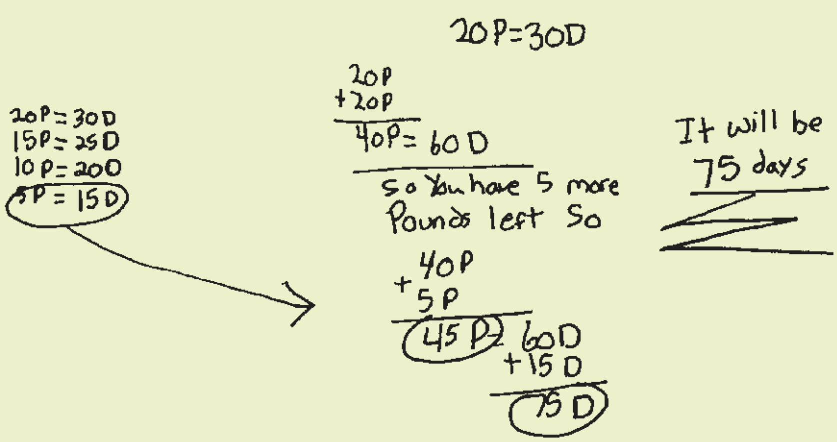
(c)

Fig. 4 The approach for this problem builds up from a baseline total.

Paul's Dog Food

Paul's dog eats 20 pounds of food in 30 days. How long will it take Paul's dog to eat a 45 pound bag of dog food? Explain your thinking.

(a)



(b)

When proportional problems contain multiplicative relationships that are easy to identify, in other words, when one element is a multiple of a corresponding element, the factor-of-change strategy is an intuitive method that students might use.

The Apple Packing problem in **figure 3a** identifies the factor of change between the two ratios as 4 (2 bushels to 8 bushels). The student then multiplies the number of boxes in the first ratio (3) by the factor of change to get the new number of boxes (3 boxes × 4 = 12 boxes.) (See **fig. 3b**.)

TRANSITIONAL METHODS

Many students intuitively try to solve proportional reasoning problems using transitional strategies. The equivalent-fraction strategy links student thinking about fractions with the concept of ratios, which involves a relationship that consists of two quantities. This method is a transitional step between a student's knowledge of fractions and understanding proportions. A student omits the label from the quantities of the ratio and focuses only on the numbers. Prior knowledge of fractions is then used to find a common denominator and compare the two numerators (VMP pilots 2006–2007). This method is effective in solving a proportion, but dropping the labels may cause problems when determining the final answer.

For example, the student work in **figure 2c** sets up the ratios without the labels, finds a common denominator of 15, and indicates that Big Horn Ranch is the correct solution, since 10/15 is greater than 9/15. Had the student included the labels that are associated with each quantity, 10 horses for every 15 acres, compared with 9 horses for every 15 acres, the smaller fraction might have been chosen as the correct solution.

Building up (or breaking down)

is another transitional method that students use to solve a proportion problem. As found in **figure 3c** and **figure 4b**, this strategy uses repeated addition or additive reasoning instead of multiplication to build up the quantities in one ratio to equal the corresponding quantities in the second ratio. Students begin by using additive reasoning to solve the problem.

As they continue to develop strategies, they move toward multiplicative reasoning, an important component of proportional reasoning. **Figure 3c** shows that 3 boxes and 2 bushels are added to each increasing quantity to build it up. Keep in mind, however, Lamon's statement that "successes with additive strategies do not necessarily encourage the exploration and adoption of more efficient strategies and should not be interpreted as proportional reasoning" (Lamon 2005, p. 100). Steinthorsdottir notes: "Proportional reasoning is multiplicative and therefore the transition from build up strategies to multiplicative strategies is considered to be a benchmark of development" (2005, p. 227).

In Paul's Dog Food problem, the work in **figure 4b** begins using the building-up strategy, adding 20P + 20P = 40P, where P represents pounds, to be equivalent to 60D, where D represents days. It shows that 5 more pounds are left. The student now calculates how many days it will take the dog to eat the remaining 5 pounds of food. The student incorrectly reverts to additive reasoning and subtracts 5 repeatedly from the values given in the problem, 20 pounds and 30 days, until he incorrectly determines that 5 pounds is equal to 15 days. His final incorrect solution states that 40P + 5P= 45P is equal to 60D + 15D = 75D.

Fig. 5 Doubling the numerator and denominator (b) and multiplication and division (c) are used to solve this problem.

If 3 loaves of bread make 20 sandwiches, how many loaves of bread are needed to make 300 sandwiches?

(a)

× 2 × 2 × 2 × 2 Subtract initial ratio

Loaves of bread	3	6	12	24	48	–3	45
Sandwiches	20	40	80	160	320	–20	300

× 2 × 2 × 2 × 2

(b)

× 10 ÷ 2 × 3

Loaves of bread	3	30	15	45			
Sandwiches	20	200	100	300			

(c)

MORE STRUCTURED METHODS

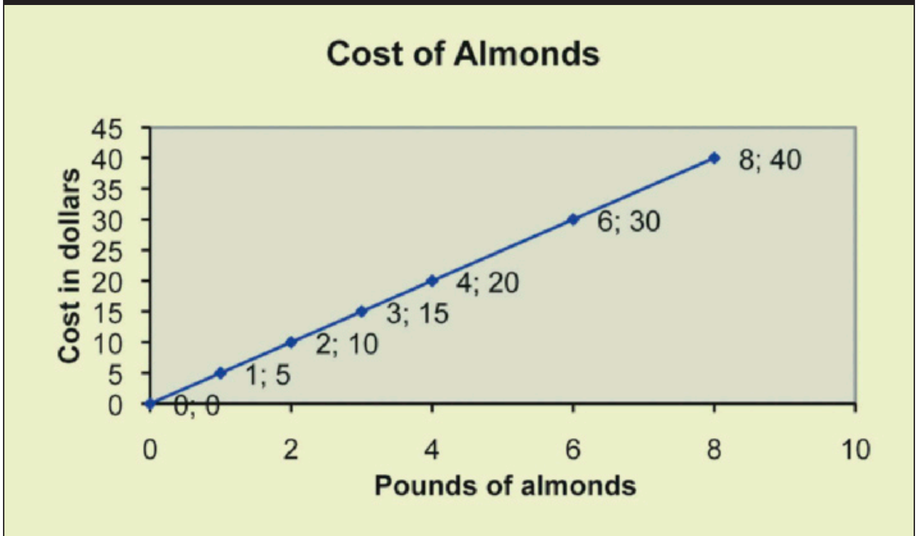
Using ratio tables and graphs are two structured methods for solving proportion problems. The ratio table may be a next step after students have worked through the building-up method. Students can use a ratio table to organize numbers and record operations and results while documenting the intermediate steps in solving a ratio problem (Middleton and van den Heuvel-Panhuizen 1995). It also supports students' proportional thinking. The equivalent ratios within the table can be generated using a multiplicative strategy.

A ratio table can be used in many ways to solve problems, which is, in part, why it is so valuable. Students can use it in whatever way makes sense to them, regardless of the number of columns in the table. **Figure 5a** offers another problem, and two ways to approach it. The strategy of doubling the numerator and denominator to solve the problem is shown in **figure 5a**. **Figure 5b** uses multiplication and division.

The ratio table is a structure that can move students to the more efficient multiplicative strategy of the factor of change. For example, a student may discover that multiplying a number by 10, dividing by 2, and multiplying again by 3 is equivalent to multiplying by 15.

Another structured tool is a graph. When using a graph to solve a proportion problem, all equivalent ratios or rate pairs lie in a straight line ($y = kx$) that passes through the origin. In the cost of almonds graph (see **fig. 6**), the points represent different quantities of almonds that can be purchased and the corresponding cost. Start with any pair, such as 2 pounds of almonds that cost $10, and graph the line and generate all other points. Then, if purchasing 6 pounds of almonds,

Fig. 6 Equivalent rate pairs of a proportion problem fall on the same line.



a student could use the graph to find the cost of \$30.00. If the dollar amount is \$20.00, the student could determine that 4 pounds of almonds could be purchased. All these ratios can be simplified to the same unit rate of \$5.00 for 1 pound of almonds. This rate also represents k, the slope of the line; when k is expressed with the denominator of 1, k is the unit rate. Note that the slope of the line is the constant of proportionality providing the multiplicative relationship between the variables. Any number of pounds of almonds x multiplied by the unit rate k in \$/lbs. will give the total cost in dollars of the almonds y, or $y = kx$.

The cross-multiplication method can also be used to solve proportion problems. However, research has shown that this particular method is "(1) poorly understood by students, (2) seldom a 'naturally generated' solution method, and (3) often used by students to avoid proportional reasoning rather than to facilitate it" (Lesh, Post, and Behr 1988, http://www.cehd.umn.edu/rationalnumberproject/88_8.html). But it does have an important role when solving proportion problems, especially when viewing the relationships between the numbers if they are not integers. Smith and colleagues note that "Once solid conceptions of proportionality have been developed, *cross multiplication* can be introduced as an efficient algorithm for solving missing-value proportion problems—especially those for which the numbers in the problem make intuitive strategies difficult or cumbersome to apply" (Smith, Boston, and Hillen 2003, p. 150).

It is important that teachers connect the standard algorithm of cross multiplication with students' prior learning so that this approach is not solely understood as a rote procedure. For example, when discussing various methods, the factor-of-change and the unit-rate methods can be linked to student understanding of how cross multiplication works. This missing-value problem was solved using the factor-of-change method:

$$\frac{24\text{ apples}}{\$6} = \frac{72\text{ apples}}{\$18}$$

In the Apple problem, the factor-of-change rate between the two ratios was 3. The ratios can be rewritten to show the factor of change within the solution:

$$\frac{24}{6} = \frac{24 \cdot 3}{6 \cdot 3}$$

The ratios are equivalent, since the numerator and the denominator are being multiplied by the same number. Cross multiplication illustrates this equivalence:

$$24 \cdot (6 \cdot 3) = 6 \cdot (24 \cdot 3)$$

The cross-multiplication method can be used to find the missing value, x:

$$\frac{24}{6} = \frac{x}{18}$$

First, rewrite the problem to show the factor of change:

$$\frac{24}{6} = \frac{x}{6 \cdot 3}$$

Then cross multiply:

$$24 \cdot (6 \cdot 3) = 6 \cdot x$$

The missing factors that are equivalent to x are 24 and 3, the same factors in the first cross multiplication example above.

The standard algorithm for cross multiplication can also be linked to the student's understanding of unit rates. The unit rate for the Apple problem

$$\frac{24\text{ apples}}{\$6} = \frac{72\text{ apples}}{\$18}$$

is 4 apples per \$1.

The ratios can be rewritten to show the unit rate within the solution:

$$\frac{6 \cdot 4\text{ apples}}{6 \cdot \$1} = \frac{18 \cdot 4\text{ apples}}{18 \cdot \$1}$$

The ratios are equivalent, since the numerator and the denominator are being multiplied by the same number. Cross multiplication illustrates this equivalence:

$$(6 \cdot 4) \cdot (18 \cdot 1) = (6 \cdot 1) \cdot (18 \cdot 4)$$

The cross-multiplication method can connect back to intuitive methods that students generate themselves when solving a proportion problem. The connection between students' intuitive methods and the standard algorithm for solving proportions has been acknowledged by several mathematics educators. "The curriculum should provide extensive opportunities over time for students to explore proportional situations concretely, and these situations should be linked to formal procedures for solving proportion problems whenever such procedures are introduced" (NRC 2001, p. 417).

CONCLUSION

The multiple ways to solve proportional reasoning problems outlined in this article should help teachers meet mathematical challenges within the middle school mathematics classroom. "The fact that a large portion of the adult population does not reason proportionally suggests that certain kinds of thinking do not occur spontaneously and that instruction needs to take an active role in facilitating thinking that will lead to proportional reasoning" (Lamon 2005, p. 10). Students may follow many paths as they become proportional reasoners. Some may learn intuitive strategies like the unit-rate and the factor-of change methods. Others may pursue more structured approaches using a ratio table or a graph. Ultimately, we want our students to be able to approach any proportional problem and solve it efficiently and accurately. For many students, the cross-multiplication algorithm will be used to solve higher-level-math problems. Linking the intuitive understanding of proportional reasoning to the standard algorithm should help students move toward a deeper understanding of proportionality and away from learning rote mathematics.

Students may follow many paths as they become proportional reasoners; some may learn intuitive strategies and others may pursue more structured approaches.

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Reflect and Discuss

Reflective teaching is a process of self-observation and self-evaluation. It means looking at classroom practice, thinking about what is done and why, and then evaluating whether it works. Collecting information about what goes on in the classroom and then analyzing and evaluating this information will allow teachers to identify and explore their practices and underlying beliefs.

The following questions are suggested prompts to help the reader reflect on the article and on how the authors' ideas might benefit your classroom practice. Readers are encouraged to reflect on the article independently as well as discuss it with your colleagues.

- How would you use any of these problems in your classroom to support a particular approach to understanding proportions?
- How might you use these problems differently in different grade levels to develop proportional reasoning?
- What is your evidence of student understanding about proportional reasoning? How does this provide information about your instruction? What is your next step?

Tell us how you used "Reflect and Discuss" as part of your professional development. The Editorial Panel appreciates the interest and values the views of those who send us their comments. Letters may be submitted to *Mathematics Teaching in the Middle School* at **mtms@nctm.org**. Please include "Readers Write" in the subject line. Because of space limitations, letters and rejoinders from authors beyond the 250-word limit may be subject to abridgment. Letters are also edited for style and content.

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