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Problems That Encourage Proportion Sense

A FEW YEARS AGO, A FORMER STUDENT, whom I will call Carol, asked me to help her review for a standardized mathematics test. She had solved some proportion problems on her own and wanted me to check her work. One of the problems was similar to the following: "John drove 60 miles in 2 hours. If he continues to drive at this same speed, how long will it take him to drive 40 additional miles?" To solve the problem, Carol set up two ratios, solved for the unknown using an application of the standard algorithm for proportions, and determined that driving the additional 40 miles would take John 3 hours (see **fig. 1**).

$$\frac{60}{40} = \frac{x}{2}$$
$$40x = 120$$
$$x = 3$$

Fig. 1 Carol's solution strategy

I suggested that Carol forget about her calculations for a moment and think about the problem directly. I asked her whether driving 40 miles would take John more or less time than driving the initial 60 miles would. She immediately replied that because 40 miles was a shorter distance than 60 miles, driving that distance would take less time, specifically, less than 2 hours. She also commented that because 40 miles was a bit more than half of 60 miles, driving that distance would take a bit more than 1 hour, which is half of the 2 hours needed to drive 60 miles. Using her proportion sense, or what others have called *qualitative-based* reasoning, Carol determined that her answer had to be between 1 and 2 hours and concluded that her initial answer of 3 hours did not make sense and could not be correct. Carol set up the proportion equation again and found that John would need 1 hour and 20 minutes to drive 40 miles. She was now confident that her answer had to be correct. Carol's use of proportion sense was essential to her understanding of the proportional situation of the time required for John to drive a certain distance.

NCTM's *Principles and Standards for School Mathematics* (2000) emphasizes the importance of developing and nurturing middle school students' proportional reasoning abilities. One of the ways to do so is to help students understand how to coordi-

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nate the various quantities in the ratios that are being compared so that relationships between these quantities can be explored and extended. The ability to reason about quantities and the various relationships that quantities share in proportional situations is what I call *proportion sense.*

This article describes four problems that encourage the use and development of proportion sense. These problems were created during a research study with prospective elementary school teachers (Billings 1998). As part of the study, the participants solved a variety of proportion problems. Excerpts from their solution strategies are presented to show how these problems created a rich environment for proportion sense to emerge and develop. Pseudonyms have been given to the participants in the study.

Using Nonnumeric Problems to Encourage Proportion Sense

WHEN SOLVING NUMERIC PROPORTION PROBLEMS, many students are so concerned with getting an answer that they fail to consider the reasonableness of their answers. Many attempt to apply the standard algorithm or some other quantitative procedure without truly understanding why the algorithm or procedure is appropriate (Cramer and Post 1993). As teachers, we face the challenge of making proportions meaningful for our students. Because students tend to "misuse" numbers by applying them to some formula, we must help them focus on underlying proportional relationships and create an environment that nurtures proportion sense. One way that we can help students cultivate proportion sense is to strip problems of numbers, that is, provide nonnumeric proportion problems, that force students to examine the relationships between variables directly.

Nonnumeric biking-speed problem

Consider the following nonnumeric proportion problem that involves two girls biking on a path:

> Catherine and Rachel like to ride their bicycles along the bike path in Forever Green Park. Today, they both started riding at the beginning of the trail; each rode continuously at a constant speed, making no stops, to the end of the trail. Rachel took more time than Catherine to reach the end of the path. Which girl was biking faster? Why? Explain your answer.

This problem context is similar to any standard problem that could be found in a chapter of a middle school textbook dealing with proportions. In this setting, however, students are asked to make a conclusion about a relationship—which girl was riding faster—rather than calculate a numerical answer—the speed or distance traveled by one or both of the girls.

The fourteen prospective teachers who participated in the study all used their proportion sense to reason that Catherine rode her bike faster than Rachel. They concluded that because the bikers did not make any stops and rode the same distance, they must have taken different amounts of time to reach the end of the path. Consequently, the girls had to be riding at different speeds. For example, Grace explained the problem this way:

> Catherine made it to the end first, so she had to have been moving faster. . . . She was able to reach the end of the path first. Rachel must have been moving slower; otherwise, she would have arrived at the same time or before Catherine.

Students often fail to consider the reasonableness of answers

Because the problem is open ended, however, it also allows students to further analyze unstated assumptions that could affect the proportional relationship. For example, Bill said that Catherine was biking faster, "assuming she didn't take a shortcut." He noted that a decrease in the total distance that Catherine was biking would affect the amount of time needed to reach the end of the trail. In a class discussion, Bill's observation offered an opportunity to discuss the role of distance in this problem. Students could see that if the girls rode two different distances, the length of time required to reach the end of the path would not be sufficient information to determine who was the faster biker. Mary found another example of an unstated assumption in this problem. She observed, "They both started riding at the beginning of the trail, but they didn't say they both started at the same time, which may mean that one may have started before the other." She realized that if the time in the problem did not mean actual time spent riding, then it could not be used as a means for comparison either.

This problem not only focuses on the relationship between time and speed but also allows students to analyze other factors, such as distance and starting time, that might affect whether the relationship is truly proportional. This seemingly simple problem provided an opportunity to reason about the relationship between variables that could both directly and indirectly affect the bikers' speeds.

Nonnumeric piano-string-vibration problem

Another example of a problem that promotes proportion sense was modified from a standard textbook problem involving piano strings.

> The frequency of vibrations of a piano string increases as the length decreases. Which piano string would vibrate more slowly, a 36-inch string or a 24-inch string? Why? Explain your answer.

Here, the statement of the problem gives numerical values, but not enough values are given to calculate an exact answer. As a result, the student must analyze the relationship between the length of the string and the frequency of the string's vibration. Again, all the prospective teachers reasoned that the 36-inch string would vibrate more slowly. Joan's response is typical of this type of reasoning:

> So it would be a 36-inch string that is vibrating more slowly because the less string, the more it vibrates. A 36-inch string would vibrate more slowly for that reason. The shorter the length, the more vibrations. The longer the length, the less vibrations.

Students must reason about relationships that exist between items

In this problem, students must focus on the underlying proportional relationship that connects the various quantities and determine how an increase or decrease in one quantity, in this instance, the length of the string, directly affects the behavior of another quantity, the frequency of the vibrations.

Nonnumeric coffee-taste problem

In addition to modifying standard textbook proportion problems, I also developed a series of nonnumeric problems. These problems show a picture of two carafes of coffee and indicate which carafe contains stronger-tasting coffee. In addition, the problems state that some change is made to the carafes, such as adding a cup of water or adding a spoonful of instant coffee. The objective is to identify, if possible, the carafe that contains the stronger-tasting coffee after the change has been made. Once again, these problems require students to reason about the relationships that exist between the amount of water or coffee and the taste of the coffee, thus cultivating students' proportion sense in a different context. See **figure 2** for an example of this type of problem.

The prospective teachers also answered this question correctly by reasoning about the effects of adding more coffee and water to the coffee mixtures in the carafes. They realized that as the amount of coffee increased, the concentration of the mixture also increased. Likewise, as the amount of water added to a mixture increased, the concentration of the mixture decreased. For example, Lucy explained the relationship as follows:

> [Carafe B] is already weaker to start off with, and then you're still going to add more water to it; you're not even adding any coffee at all. And [carafe A] is already strong, and you're just adding coffee to it, which is going to make it stronger.

Making a mathematically meaningful comparison between carafes in a different and slightly more complicated situation (see **fig. 3**) proved more challenging for the study participants. Most of the prospective teachers successfully identified the initial volume of the liquid as a necessary component in determining the relationship between the addition of more coffee and the taste of the coffee. Bill reasoned as follows:

> If they both taste the same, I would assume they would both have the same ratio of water to coffee, [and you are] adding one spoon of coffee to each. In carafe A, it would be diluted more. In carafe B, there would be less water to dilute the coffee; therefore, it would be stronger.

Misconceptions about the proportional relationships also emerged. Several prospective teachers reasoned incorrectly, as Grace did when she said, "The coffee tastes the same; add coffee to both. So

Below it is indicated which carafe contains the stronger coffee. Determine which carafe will contain the stronger coffee after the alterations have been made. Explain how you came to your answer.

Carafe B contains weaker coffee than carafe A. Add one spoon of instant coffee to carafe A and one cup of water to carafe B.

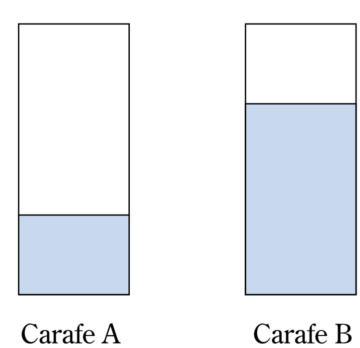


Fig. 2 Nonnumeric coffee-taste problem 1

it's always the same . . . because you're adding the same amount to each and they started out the same." Grace, like several others, was operating under the assumption that in adding the same amount of coffee to both carafes, the relative relationship of the taste also remained the same. These students disregarded the initial volume of liquid in the carafes, not realizing that this variable was essential in analyzing the situation. They realized that an increase in the amount of coffee caused an overall increase of stronger-tasting coffee in each carafe, but they failed to compare the overall strength of the coffee *between* carafes, an important component in this proportional situation. Their reasoning appears to focus on doing the "same thing" to both carafes. This reasoning illustrates the teachers' misunderstanding of the notion that simultaneously increasing the value of one quantity in a pair of corresponding ratios can affect the relationship between the two ratios. Their reasoning can be summarized as follows:

If

$$\frac{a}{c} = \frac{b}{d} \qquad (c, d \neq 0),$$

then

$$\frac{a+1}{c} = \frac{b+1}{d} \qquad (c, d \neq 0),$$

when the values of *a, b, c,* and *d* are unknown. However, altering corresponding values in a ratio changes not only the ratio itself but also the relationship between the ratios.

Below it is indicated which carafe contains the stronger coffee. Determine which carafe will contain the stronger coffee after the alterations have been made. Explain how you came to your answer.

Carafe A and carafe B contain coffee that tastes the same. Add one spoon of instant coffee to both carafe A and carafe B.

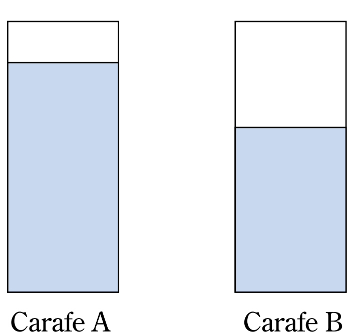


Fig. 3 Nonnumeric coffee-taste problem 2

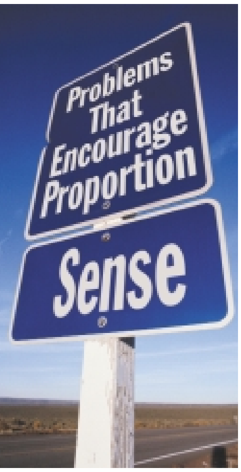
This nonnumeric proportion situation exposed certain assumptions that the teachers made about the relationships that exist between the quantities in a proportional setting. If the problem had involved numerical quantities and asked the teachers to calculate a numerical answer rather than to directly reflect on the relationship between the quantities in the ratios, this fundamental misunderstanding may not have emerged. When such misconceptions emerge as students solve this type of nonnumeric problem, the opportunity arises to discuss the importance of also comparing the relationship between the new ratios, such as, in this instance, comparing the altered carafes of coffee. The discussion will help students strengthen their proportion sense.

Nurturing Proportion Sense in a Variety of Proportion Situations

Nonnumeric problems offer a different type of context

THE FOUR NONNUMERIC PROBLEMS presented in this article were all designed to deepen students' understanding of the underlying concepts that influence the relationships in proportional situations. Because nonnumeric problems demand that students examine relationships, having students solve these types of problems, especially when the concept of proportions is being introduced, can be beneficial. Many traditional proportion word problems can be altered to focus primarily on the underlying proportional relationship, as was done in the bicycle and piano-strings problems. Simply modify the proportion problem in such a way that some of or all the numerical quantities are deleted but the relationship between the variables is still clear, then ask a question that requires the students to focus on the proportional relationship inherent in the problem.

Nonnumeric problems offer a different type of context in which to examine the underlying relationships that connect the variables in proportional situations. If your students cannot relate to carafes of coffee, alter the context to something that has more meaning to them, such as pitchers of hot chocolate or lemonade. These problems can also be extended to include additional scenarios in which students are asked to contrast the results of adding quantities of some type of variable to various mixtures, then to compare concentrations.



By presenting students with different nonnumeric proportion problems, especially as they are introduced to proportions, we will help them focus on the underlying relationships and anticipate how an increase or decrease in one variable affects another variable, as well as alters the ratio. Exposing students to nonnumeric proportion situations is essential in helping them develop proportion sense.

Using Numeric Problems for Proportion Sense

DEVELOPING PROPORTION SENSE IS NOT LIMited to these types of nonnumeric problems. Most of the proportion problems that students face in their studies will involve numbers. Teachers can build students' proportion sense and understanding of proportions in numeric situations. One way to incorporate numeric problems is to first discuss and solve a nonnumeric problem, such as the bicycle or piano-string problems presented previously. Then, extend the discussion by posing a similar numeric problem. This approach allows students to apply their developing proportion sense in a concrete, numeric setting.

We can also encourage our students to use proportion sense and focus first on underlying relationships in numeric proportion situations. For example, the problem that introduced this article asks students to determine the length of time John needs to drive 40 miles given that he has driven 60 miles in 2 hours. Instead of immediately encouraging the use of some quantitative strategy, such as the standard algorithm or some other solution strategy, begin by asking the students, "Do you think John will drive the 40 miles in more or less than 2 hours? Why do you think so?" This type of question communicates to our students that we want them to make sense of the problem and their proposed answers. In addition, the question emphasizes the importance of reflecting about the nature of the underlying relationships in the problem first. After students have concluded that driving the 40 miles would take less time because the distance is shorter, you might ask, "Could anyone make a reasonable estimate about how long John would take to drive 40 miles? Why is your estimate reasonable?" After reflecting about the relationships, students could then be asked to think about what solution strategy might be appropriate to solve this problem and to determine the exact amount of time required to drive the 40 miles.

Summary

RESEARCH HAS SHOWN THAT EMPHASIS ON THE standard algorithm for solving proportions—equating two ratios with one unknown, cross-multiplying, and solving for the unknown—has limitations in encouraging students to reason proportionally (Post, Behr, and Lesh 1988). However, this standard algorithm is a procedure that middle school students are taught in a traditional curriculum. Although this algorithm can be useful and efficient for solving proportion problems, it may suggest to students that they simply memorize the procedure; they do not necessarily have to reason about the various quantities that constitute the ratios, nor directly reason about how one variable is related to another. In contrast, when students are expected to reason about the relationships between the variables and use proportion sense, they can then begin to make more sense of the standard algorithm and other strategies for solving numeric proportion situations.

Most of the proportion problems that students encounter are quantitative, and their knowing how to solve proportion situations numerically is important; but students must also be able to reason directly about the relationships that exist between the quantities in these situations. As students solve numeric proportion problems, teachers must ask questions that will help them continue to reflect about the underlying relationships that exist in the problem. If we encourage this type of reflection, students will begin to form the habit of asking themselves questions about underlying proportional relationships, thus strengthening their proportion sense. Students who actively use proportion sense can make sense of the relationships in the problem and verify that their final answers make sense after applying some quantitative strategy, such as the standard algorithm, or unit-rate method. Emphasizing proportion sense can also help students make sense of, and draw mathematical meaning from, ratio and proportion situations. As their proportion sense grows, students deepen their understanding of ratios and proportions and become better mathematical thinkers.

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